1. General assumptions

Throughout, we take $x \diamond y = x + f(h)$ where $h := y - x$. Our carrier set will always be the free abelian group on countably many generators A. We will also fix a well-ordering of A, with order type ω.

For use later, we let p, q, r, s be four of the independent generators of A.

2. Extending partial examples for Equation 73

Equation 73 is $x = y \diamond (y \diamond (x \diamond y))$. Its linearization is

$$\begin{aligned} x &= y \diamond (y \diamond (x + f(h))) \\ &= y \diamond (y + f(f(h) - h)) \\ &= x + h + f^2(f(h) - h). \end{aligned}$$

Thus, the corresponding functional equation is

$$f^2(f(h) - h) = -h.$$

Define $\mathscr{E}$ to be the collection of sets $E \subseteq A^2$ satisfying the following conditions.

(1) E is finite.
(2) E is a function.
(3) E satisfies the functional equation at $h = 0$.
(4) If $(a, b), (c, d) \in E$ are distinct pairs, then $b - a \neq d - c$.
(5) If $(a, b), (b - a, c) \in E$, then $(c, -a) \in E$. (This guarantees that the functional equation holds when $h = a$.)

Partially order $\mathscr{E}$ by set inclusion.

Lemma 2.1. *Given any $E \in \mathscr{E}$ and any $a \in A$, there is an extension of E in $\mathscr{E}$ where the functional equation holds for a.*

Proof. If $f(a)$ and $f(f(a) - a)$ are both defined by E, there is nothing to do.

If $f(a)$ is not defined (so $a \neq 0$ by (3)), then letting b be any generator of A not appearing in E, the set $E \cup \{(a, b)\}$ still satisfies the first four conditions and $f(a)$ is now defined. Condition (5) will also be satisfied unless $a = y - x$ for some pair $(x, y) \in E$ (which is unique by (4)), in which case we must also adjoin $(b, -x)$. Now, $E \cup \{(a, b)\} \cup \{(b, -x)\}$ again satisfies the first four conditions. The fifth also holds, by a quick check.

So, we reduce to the case where $b := f(a)$ is defined but $f(b - a)$ is not. (Hence $a, b - a \neq 0$.) Taking c to be any new generator, the set $E' := E \cup \{(b - a, c), (c, a)\}$ satisfies all conditions. (To check condition (5), it suffices to check that $c - b + a$ and $a - c$ are not the first coordinates of any ordered pairs in E', and c is not the difference of entries in any ordered pair. This is clear since $b - a \neq 0$.) $\square$

As before, this means that any element of $\mathscr{E}$ can be extended to a full function satisfying the functional equation for Equation 73.

3. 73 does not imply 99

Equation 99 is $x = x \diamond ((x \diamond x) \diamond x)$. Its linearization is

$$\begin{aligned} x &= x \diamond ((x + f(0)) \diamond x) \\ &= x \diamond (x + f(0) + f(-f(0))) \\ &= x + f(f(0) + f(-f(0))). \end{aligned}$$

The family $\{(0, p), (p, q), (q, 0), (-p, r), (p+r, s), (s, p)\}$ fails the new functional equation, but is in $\mathscr{E}$, hence extends to a counterexample for the implication from Equation 73 to Equation 99.

4. 73 does not imply 203

Equation 203 is $x = (x \diamond (x \diamond x)) \diamond x$. Its linearization is

$$\begin{aligned} x &= (x \diamond (x + f(0))) \diamond x \\ &= (x + f^2(0)) \diamond x \\ &= x + f^2(0) + f(-f^2(0)). \end{aligned}$$

The family $\{(0, p), (p, q), (q, 0), (-q, r), (r, -q), (q+r, s), (s, q)\}$ fails the new functional equation, but is in $\mathscr{E}$.

5. 73 does not imply 255

Equation 255 is $x = ((x \diamond x) \diamond x) \diamond x$. Its linearization is

$$\begin{aligned} x &= ((x + f(0)) \diamond x) \diamond x \\ &= (x + f(0) + f(-f(0))) \diamond x \\ &= x + f(0) + f(-f(0)) + f(-f(0) - f(-f(0))). \end{aligned}$$

The family $\{(0, p), (p, q), (q, 0), (-p, r), (-p - r, s), (p + r + s, t), (t, p + r)\}$ fails the new functional equation, but is in $\mathscr{E}$.

6. 73 does not imply 4380

Equation 4380 is $x \diamond (x \diamond x) = (x \diamond x) \diamond x$. Its linearization is

$$x + f^2(0) = x + f(0) + f(-f(0)).$$

The family $\{(0, p), (p, q), (q, 0), (-p, r), (p+r, s), (s, p)\}$ fails the new functional equation, but is in $\mathscr{E}$.

7. Undone stuff

By specialization, we see that 73 also does not imply 125, 229, or 4335. We finish this note by showing, conversely, that 229 does not imply 73, thus completely handling 73.

Equation 229 is $x = (y \diamond (y \diamond x)) \diamond y$. Its linearization is

$$\begin{aligned} x &= (y \diamond (x + h + f(-h))) \diamond y \\ &= (x + h + f^2(-h)) \diamond (x + h) \\ &= x + h + f^2(-h) + f(-f^2(-h)). \end{aligned}$$

Replacing h with $-h$, the corresponding functional equation is

$$f(-f^2(h)) = h - f^2(h).$$

Define $\mathscr{F}$ to be the collection of sets $F \subseteq A^2$, subject to the following conditions.

(1) F is finite.
(2) F is an injective function.
(3) F satisfies the functional equation at 0.
(4) If $(a, b), (b, c) \in F$, then $(-c, a - c) \in F$.

Order the set $\mathscr{F}$ as usual, and note that the set

$$F_0 := \{(0, p), (p, q), (q, 0), (-q, -q), (-p, -p + q)\}$$

does satisfy condition (4), so $\mathscr{F}$ is nonempty. We have the following lemma.

Lemma 7.1. *Given any $F \in \mathscr{F}$ and any $a \in A$, there is an extension of F in $\mathscr{F}$ where the functional equation holds for a.*

Proof. We quickly reduce to the case where $b := f(a)$ is defined, but $f(b)$ is not. In particular, $a \neq 0$. By taking $F' := F \cup \{(b, c), (-c, a - c)$, where c is a new generator, we are done once we verify the conditions for belonging to $\mathscr{F}$.

The main thing to check is the last condition, where we need $f(c)$ and $f(a - c)$ to be undefined, and we need $-c$ not in the image. These are clear, since $a \neq 0$. □

Finally, we need a starting "seed" in $\mathscr{F}$ where 73 fails. Taking t, u, v, w to be new independent generators, $F_0 \cup \{(t, u), (u - t, v), (v, w), (-w, u - t - w)\}$ works.