Stochastic integrals in Lean

Weakly meeting

30/01/2026

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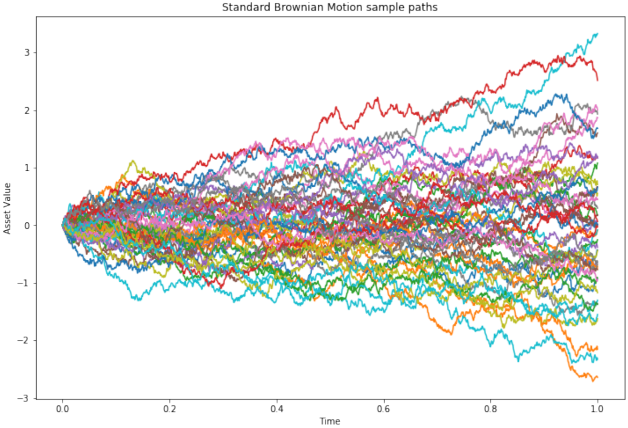
Brownian motion



Formalized.

Stochastic integrals

Goal: Ito's lemma in Mathlib.

Most of the work will be to define stochastic integrals.

$$\int_0^t X_s \, dY_s$$

For what kind of $X : T \to \Omega \to E$ and $Y : T \to \Omega \to F$?

Stochastic integrals

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$$\int_0^t X_s \, dY_s$$

For what kind of $X : T \to \Omega \to E$ and $Y : T \to \Omega \to F$?

- Y is a semimartingale: cadlag local martingale M + finite variation process A
- X is a predictable process in some L^2 space

$$\int_0^t X_s \, dY_s = \int_0^t X_s \, dM_s + \int_0^t X_s \, dA_s$$

Finite variation integral

Finite variation process A. For each ω, $A_\cdot(\omega)$ is a finite variation function.

- Defines a vector measure $dA(\omega)$ on T (Sébastien Gouëzel is working on this).
- $\int_0^t X_s \, dA_s$ is the integral of X over time with respect to this vector measure (for each ω). (Yoh Tanimoto and Oliver Butterley have PRs about that integral)

Martingale integral

Cadlag local martingale M. What is $\int_0^t X_s \, dM_s$?

- $\|M\|^2$ is a cadlag local submartingale
- M has a quadratic variation process $\langle M \rangle$ (cadlag, non-decreasing) built from Doob-Meyer decomposition of $\|M\|^2$.
- $\langle M \rangle(\omega)$ defines a measure $d\langle M \rangle(\omega)$ on T (Stieltjes measure, in Mathlib)
- We can define an integral for simple predictable processes.
- It extends to a larger class of predictable processes in $L^2(\Omega \times T, \mathcal{P}, Pd\langle M \rangle)$ by density and Ito isometry.

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2 Preliminaries

Continuous time definitions and results

- Right-continuous filtration, right continuation
- Predictable sigma-algebra and processes
- Simple predictable processes

- Lemmas about uniform integrability
- Optional sampling in continuous time (martingales ok, submartingales WIP)
- Doob's inequalities in continuous time (WIP)

Locally

- Localizing sequences of stopping times (monotone increasing to infinity)
- Local property: a property on processes holds locally if there exists a localizing sequence such that the property holds for the stopped processes $X^{\tau_n}\mathbb{I}_{\tau_n>0}$.
- Local (sub)martingales (locally martingale+cadlag)
- Stable classes of processes under localization

Measurability of stopping times

Definition (Hitting time)

For $X : T \to \Omega \to E$ a stochastic process, B a subset of E and $t_0 \in T$, the hitting time of X in B after t_0 is the random variable $\Omega \to T \cup \{\infty\}$ defined by

$$\tau_{B,t_0}(\omega) = \inf\{t \in T \mid t \geq t_0,\ X_t(\omega) \in B\}\ ,$$

in which the infimum is infinite if the set is empty.

Theorem

If $X : T \to \Omega \to E$ is a progressively measurable process with respect to a right-continuous filtration and B is a Borel-measurable subset of E, then the hitting time of X in B is a stopping time.

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Quadratic variation

How do we define the quadratic variation process $\langle M \rangle$ for a cadlag local martingale?

$\|M\|^2$ is a local submartingale: use the Doob-Meyer decomposition theorem.
$\|M\|^2 = \langle M \rangle +$ martingale.

Doob-Meyer decomposition

Theorem (Doob-Meyer)

Let $S = (S_t)_{0 \leq t \leq T}$ be a cadlag submartingale of class D. Then, S can be written in a unique way in the form $S = M + A$ where M is a cadlag martingale and A is a predictable increasing process starting at 0.

Theorem (Local Doob-Meyer)

An adapted process X is a cadlag local submartingale iff $X = M + A$ where M is a cadlag local martingale and A is a predictable, cadlag, locally integrable and increasing process starting at 0. The processes M and A are uniquely determined by X a.s.

- Doob-Meyer: basically not started (some auxiliary results done, like a Komlos lemma)
- Local submartingales are locally of Class D: mostly done

Simple processes

Definition (Simple process)

Let $(s_k < t_k)_{k\in\{1,\ldots,n\}}$ be points in a linear order T with a bottom element 0. Let $(\eta_k)_{0\leq k\leq n}$ be bounded random variables with values in a normed real vector space E such that η_0 is $\mathcal{F}_0$-measurable and η_k is $\mathcal{F}_{s_k}$-measurable for $k \geq 1$. Then the simple process for that sequence is the process $V : T \to \Omega \to E$ defined by

$$V_t = \eta_0 \mathbb{I}_{\{0\}}(t) + \sum_{k=1}^{n} \eta_k \mathbb{I}_{(s_k,t_k]}(t) .$$

Elementary stochastic integral

Definition (Elementary stochastic integral)

Let $V \in \mathcal{E}_{T,E}$ be a simple process and let X be a stochastic process with values in a normed space F. Let B be a continuous bilinear map from $E \times F$ to another normed space G. The *general elementary stochastic integral* process $V \bullet_B X : T \to \Omega \to G$ is defined by

$$(V \bullet_B X)_t = \sum_{k=1}^{n} B(\eta_k, X_{t_k}^t - X_{s_k}^t) \; .$$

Two important special cases:

- $E = L(F, G)$ and $B(L, x) = L(x)$ the evaluation map
- $E = \mathbb{R}$ and $B(r, x) = rx$ the scalar multiplication

Next steps

What we know:

- Elementary integral is linear, associative

What we need:

- Relation with martingales
- Extension to larger classes of integrands (density + isometry)
- Properties of the integral obtained

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4 Project organization

Project organization

Remarks:

- We need to flesh out the blueprint near the end (and replace continuous martingales by cadlag).

Questions:

- Do we focus on one subgoal at a time, or work in parallel on different subgoals?
- How atomic are the tasks? Do we assign mini-projects to teams (ideally experienced + novice)?
- Bottom-up or more distributed?
- How do we make sure we reach Mathlib?
- Do we switch to LeanArchitect? (latex in lean files, automatic dependency graph)

Mini-projects

(or not-so-mini projects)

- Doob's inequalities in continuous time (in progress)
- Optional sampling, stopping and convergence theorems in continuous time
- Local submartingales are locally of Class D (mostly done)
- Début theorem (in progress? stalled?)
- Existence of cadlag modifications for submartingales
- Doob-Meyer decomposition theorem
- Simple processes and Ito isometry (assuming quadratic variation)

References

- Almost Sure blog. Good for preliminaries. Different integral construction.
- Stochastic Calculus and Applications lecture notes.
- *Foundations of Modern Probability*, Kallenberg. General reference. All in $\mathbb{R}$.
- *Brownian Motion, Martingales, and Stochastic Calculus*, Le Gall.
- *Stochastic Equations in Infinite Dimensions*, Da Prato and Zabczyk. Hilbert-valued processes.
- *Stochastic Integration*, Metivier, Pellaumail. Perhaps most general construction of stochastic integrals?