What's missing in Mathlib?

Yury Kudryashov,
Texas A&M University

Feb 7, 2024
Day 714 of Russian full-scale invasion of Ukraine
Lean Seminar, Rutgers University

Some theorems and definitions in order they were added to Mathlib

- Banach fixed-point theorem (Alistair Tucker, 2018/12/26)
- Banach open mapping theorem (Sébastien Gouzel, 2019/04/30)
- Rolle Theorem (YK, 2019/12/23)
- Uniqueness of solutions of ODEs (YK, 2020/01/14)
- FTC-1 (YK, 2020/04/06)
- Composition of analytic functions is analytic (SG, 2020/04/17)
- Inverse function theorem (YK, 2020/04/28)
- FTC-2 (Benjamin Davidson, 2021/01/03)
- Urysohn's Lemma (YK, 2021/04/04)
- Existence of solutions of ODEs (YK, 2021/11/04)
- Maximum modulus principle (YK, 2022/02/27)
- The map $([\cdot])$ is injective (YK, 2022/10/01)
- R_1 (preregular) topological space (YK, 2024/01/29)

Missing tactics/features

Nonzero
- should handle goals like $3x^2y \neq 0$;
- could replace custom code in `field_simp`;

Positivity
- handle $x \neq 0 \Rightarrow -x^2 < 0$;
- more plugins (e.g., for `Int.fract`);
- an attribute that creates a plugin automatically.

Rewriting inside `Filter.Eventually` Make some/all of `positivity`, `gcongr`, `grw` (not yet in Mathlib) work with goals like $f \leq^f [l] g$.

Tendsto
- prove `Tendsto` by continuity
- support `cobounded`, `atTop`, `atBot`, $\mathcal{N}[\neq]x$ etc

Asymptotics
- prove asymptotic expansion of a function
- see Isabelle for prior art
- discuss with Terence Tao before implementing

More linters
- `Decidable`/`Fintype`/`Encodable` (WIP)
- Typeclass assumptions are too strong
  - Lean 3 version by Alex J. Best.
- Unnecessary `by_cases`.

Bundled sets and functions

What doesn't work
- can't have generic theorems about composition, `map`, `comap`, forgetful functors etc;
- have to repeat lots of boilerplate code for each type;

Solution 1: common structure
- define a common structure `BundledSet` α `p`
- use `abbrev Submonoid M := BundledSet M IsSubmonoid` etc
- use typeclasses about predicates to define operations

Solution 2: more typeclasses Introduce typeclasses for lawful composition, multiplication, one, zero, intersection etc.

Simple generalizations/refactors

- upstream theorems from external projects;
- define ae filter for outer measures, generalize some lemmas;
- some basic lemmas in measure theory can be generalized to any filter with countably intersection property;
- drop `T2Space` or replace it with `R1Space` here and there;
- allow `NormedSpaces` over `NormedDivisionRings`, generalize lots of lemmas;
  - coordinate with Eric Wieser
- add one more `IsCoprime`, asking for $\forall a, a \mid b \Rightarrow a \mid c \Rightarrow a \mid 1$ instead of $\exists xy, bx + cy = 1$

Definition

A set s is a *Lindelöf set*, if any open cover of s admits a countable subcover.

- Recently added by Josha Dekker.
- Many lemmas existed before, formulated either for spaces with second-countable topology or for σ-compact spaces.
- Unify API and migrate to it.

State of the art `HasFDerivAt` is defined for a function between normed spaces, thus it doesn't work for $C^\infty(\Omega)$ or matrices (unless you fix a norm);

Proposed solution add a version of `IsLittleO` for topological vector spaces, use it to generalize `HasFDerivAtFilter` etc

- proof-of-concept exists #9675
- conflicts with existing definition of gauge

Typeclass generalizations

Geometric series rings

State of the art We prove theorems about $\sum_{k=0}^{\infty} x^k$ separately for complete normed rings and for normed fields;
as a side effect, some theorems may require that a normed field is complete, even if it's not needed.

Proposed solution Define a new typeclass saying that $\sum_{k=0}^{\infty} x^k$ converges whenever $\|x\| < 1$, merge APIs

Algebraic topology

- $H_n(S^n)$, see Shamrock-Frost/BrouwerFixedPoint;
- $\pi_n(S^n)$;
- $\pi_1(S^2 \setminus s)$, where s is finite;
- universal cover;
- lift of a map to a covering space.

Manifolds

- Riemann sphere, see girving/ray.
- `AddCircle`
- more generally, quotient by a $\mathbb{Z}$-lattice
- submanifolds
- generalize smooth vector bundles to smooth bundles
- Riemannian metric
  - a proof-of-concept exists in Lean 3, see also Zulip.

Integrability

- Peano's existence theorem
- Cauchy-Kovalevskaya theorem
- Frobenius theorem
- Cartan' prolongation, Cartan-Kuranisi-Rashevsky theorem

de Rham cohomologies

- redefine topology on continuous multilinear maps (WIP);
- vector bundle of continuous multilinear maps (WIP);
- same for continuous alternating maps;
- (re)define exterior product;
- exterior derivative;
- $d(f^*\omega) = f^*(d\omega)$;
- $d(\omega \wedge \eta)$, $d(f\omega)$
- $d^2 = 0$

Single variable complex analysis

See PNT+

- Residue theorem
- Hurwitz's theorem
- Montel's theorem
- Riemann mapping theorem
- Uniformization theorem

Multivariable complex analysis, see girving/ray

- Cauchy integral formula in a polydisc.
- Hartogs's theorem on separate holomorphicity.
- Hartog's extension theorem.
- Schwarz lemma in higher dimension.
- Reinhardt domain, log-convex domain
- Holomorphically convex hull
- Stein manifolds

Fixed point and related theorems

- Brouwer fixed point theorem, see Shamrock-Frost/BrouwerFixedPoint;
- Kakutani fixed-point theorem;
- existence of Nash equilibrium;
- Schauder fixed-point theorem;
- Borsuk-Ulam theorem.

Lifts of circle self-maps $f(x+1) = f(x) + 1$

- monotone;
- strictly monotone;
- monotone continuous;
- bijective;
- C^r smooth diffeomorphisms;
- …with a break point;
- …with a critical point.

Circle self-maps $f \colon S^1 \to S^1$

- define all the same types;
- relate to lifts;
- rotation number.

- Denjoy's theorem
- Denjoy's example
- Herman-Yoccoz theorem
- Renormalization operators, their properties
- Maps with breaks, critical maps
- Random dynamics

Local normal forms of vector fields and self-maps

- Hartman-Grobman theorem
- Stable manifold theorem (Hadamard-Perron)
- finitely smooth local normal forms;
- Poincaré-Dulac normal form;
- Sternberg linearization theorem;
- and many more