Seymour Contributions

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§1 A Blueprint for Cographicness

Lemma 1.1 (Row space of a standard representation)
Let X and Y be disjoint finite sets and let

$$B \in \mathbb{F}_2^{X\times Y}.$$

Consider the matrix

$$A := [\,\mathbf{1}_x \mid B\,] \in \mathbb{F}_2^{X\times(X\cup Y)},$$

where the columns are indexed by $E := X \cup Y$ and the rows by X. Then the row space of A is

$$\mathrm{row}(A) \;=\; \{\,(u,\, uB) \mid u \in \mathbb{F}_2^X\,\} \;\subseteq\; \mathbb{F}_2^X \oplus \mathbb{F}_2^Y \cong \mathbb{F}_2^E.$$

Proof. The x-th row of A is $(e_x, B_{x,*})$, where e_x is the standard basis vector in $\mathbb{F}_2^X$ and $B_{x,*}$ is the x-th row of B. A general linear combination of the rows is therefore

$$\sum_{x\in X} u_x(e_x, B_{x,*}) = (u,\, \sum_{x\in X} u_x B_{x,*}) = (u,\, uB),$$

where $u = (u_x)_{x\in X} \in \mathbb{F}_2^X$. Conversely, every pair (u, uB) arises in this way, so these are exactly the row vectors. □

Lemma 1.2 (Orthogonal complement of a standard row space)
Let $A = [\mathbf{1}_x \mid B]$ be as in Lemma 1.1, and let

$$U := \operatorname{row}(A) \subseteq \mathbb{F}_2^{X \cup Y}.$$

Then the orthogonal complement of U is

$$U^{\perp} = \{\, (bB^{\mathsf{T}},\, b) \mid b \in \mathbb{F}_2^{Y} \,\}.$$

Equivalently, if $B^* := -B^{\mathsf{T}}$, then

$$U^{\perp} = \{\, (bB^{*},\, b) \mid b \in \mathbb{F}_2^{Y} \,\}.$$

Proof. Write vectors in $\mathbb{F}_2^{X \cup Y}$ as pairs (a, b) with $a \in \mathbb{F}_2^{X}$ and $b \in \mathbb{F}_2^{Y}$. By Lemma 1.1, any element of U has the form (u, uB) with $u \in \mathbb{F}_2^{X}$. The orthogonality condition $(a, b) \in U^{\perp}$ means

$$0 = (a, b) \cdot (u, uB) = a \cdot u + b \cdot (uB) = a \cdot u + (bB^{\mathsf{T}}) \cdot u = (a + bB^{\mathsf{T}}) \cdot u$$

for all $u \in \mathbb{F}_2^{X}$. Hence we must have $a = bB^{\mathsf{T}}$, and then

$$U^{\perp} = \{\, (bB^{\mathsf{T}}, b) \mid b \in \mathbb{F}_2^{Y} \,\}.$$

Over $\mathbb{F}_2$ we have $-1 = 1$, so $B^* = -B^{\mathsf{T}} = B^{\mathsf{T}}$, yielding the alternative description. □

Lemma 1.3 (Row space of the dual standard matrix)
With B and $B^* = -B^{\mathsf{T}}$ as above, define

$$A^* := [\, \mathbf{1}_y \mid B^* \,] \in \mathbb{F}_2^{Y \times (X \cup Y)}.$$

Then

$$\operatorname{row}(A^*) = U^{\perp},$$

where $U = \operatorname{row}(A)$ and $U^{\perp}$ is given by Lemma 1.2.

Proof. The y-th row of A^* is $(e_y, B^*_{y,*})$ with $e_y \in \mathbb{F}_2^{Y}$. A general linear combination of the rows is

$$\sum_{y \in Y} b_y (e_y, B^*_{y,*}) = (b,\, bB^*),$$

where $b = (b_y)_{y \in Y} \in \mathbb{F}_2^{Y}$. Thus

$$\operatorname{row}(A^*) = \{\, (b, bB^*) \mid b \in \mathbb{F}_2^{Y} \,\}.$$

Identifying $\mathbb{F}_2^{X \cup Y}$ as $\mathbb{F}_2^{X} \oplus \mathbb{F}_2^{Y}$ with coordinates ordered as (X, Y), this is exactly the set

$$\{\, (bB^*, b) \mid b \in \mathbb{F}_2^{Y} \,\},$$

which coincides with $U^{\perp}$ by Lemma 1.2. □

Lemma 1.4 (Dual vector matroid via orthogonal complement)
Let A and A' be matrices over a field F with the same column index set E, and suppose

$$\mathrm{row}(A') = \mathrm{row}(A)^{\perp} \subseteq F^{E}.$$

Let $M(A)$ and $M(A')$ be the vector matroids represented by A and A'. Then

$$M(A') = M(A)^{*}.$$

Proof. Let $F \subseteq E$.

(⇒) Suppose F is dependent in $M(A)$. Then there exists a nonzero vector $c \in F^{F}$ such that $A_F c = 0$. Extend c by zero outside F (still denoted c). The condition $Ac = 0$ means each row r of A satisfies $r \cdot c = 0$, hence $c \in \mathrm{row}(A)^{\perp} = \mathrm{row}(A')$. Write

$$c = \sum_i \lambda_i r_i',$$

where the r_i' are rows of A' and not all λ_i are zero. For every $e \in E \setminus F$ we have $c_e = 0$, so

$$\left(\sum_i \lambda_i r_i'\right)\big|_{E\setminus F} = 0.$$

Hence the rows of A' indexed by $E \setminus F$ admit a nontrivial linear combination giving the zero row, so $E \setminus F$ is dependent in $M(A')$.

(⇐) The same argument with A and A' interchanged, using $\mathrm{row}(A) = (\mathrm{row}(A')^{\perp})$, shows that if $E \setminus F$ is dependent in $M(A')$, then F is dependent in $M(A)$.

Thus

$$F \text{ dependent in } M(A) \iff E \setminus F \text{ dependent in } M(A'),$$

which is the defining property of duality. □

Theorem 1.5 (Dual of standard representation corresponds to dual matroid)
Let M be a binary matroid on ground set $E = X \cup Y$, with standard representation B so that

$$A = [\,\mathbf{1}_X \mid B\,].$$

Let $B^* := -B^{\mathsf{T}}$ and

$$A^* := [\,\mathbf{1}_Y \mid B^*\,].$$

Then $M(A^*) = M(A)^* = M^*$.

Proof. By Lemma 1.1 and Lemma 1.2, if $U = \mathrm{row}(A)$ then $U^{\perp}$ has the form

$$U^{\perp} = \{\,(bB^*, b) \mid b \in \mathbb{F}_2^{Y}\,\}.$$

By Lemma 1.3, we have

$$\mathrm{row}(A^*) = U^{\perp} = \mathrm{row}(A)^{\perp}.$$

Therefore, by Lemma 1.4, the column-matroid $M(A^*)$ is the dual of $M(A)$:

$$M(A^*) = M(A)^* = M^*.$$

□

Lemma 1.6

The dual matroid of a regular matroid is also a regular matroid.

Proof. Let M be a regular matroid. We wish to show that M^* is also regular.

Take a standard $\mathbb{Z}_2$-representation matrix B of M. By Lemma 34, since M is regular, there exists a TU signing B' of B: B' is a matrix over $\mathbb{Q}$ that is TU, and $|B'(i,j)| = B(i,j)$ for all entries. So M is represented (over $\mathbb{Q}$) by a TU matrix B' whose pattern of zero and non-zero entries is exactly that of B.

From Theorem 1.5, if a matroid M has standard representation matrix B, then its dual M^* has the standard representation matrix $B^* = -B^\intercal$. The TU signing of this dual standard matrix, $(B')^* = -(B')^\intercal$, preserves total unimodularity, so $(B')^*$ is a TU matrix whose support is exactly B^*.

Since we have just exhibited a TU signing of M^* (i.e., $(B')^*$), the dual matroid M^* is regular by Lemma 34. □

Theorem 1.7

Every cographic matroid is regular.

Proof. We know that all graphic matroids are regular by Theorem 70. Recall that we say a matroid is cographic if its dual is graphic. So it suffices to show regularity is preserved under duals, which we showed in Lemma 1.6. □