PART I. AUTOMATA ON INFINITE WORDS

Notation

Throughout this chapter A denotes a finite alphabet and A^*, respectively A^ω, denote the set of finite words, resp. the set of ω-sequences (or: *ω-words*) over A. An ω-word over A is written in the form $\alpha=\alpha(0)\alpha(1)\ldots$ with $\alpha(i)\in A$. Let $A^\infty=A^*\cup A^\omega$. Finite words are indicated by u, v, $w,\ldots$, the empty word by ε, and sets of finite words by U, V, $W,\ldots$ Letters α, $\beta,\ldots$ are used for ω-words and L, $L',\ldots$ for sets of ω-words (i.e., *ω-languages*). Notations for segments of ω-words are

$$\alpha(m,n):=\alpha(m)\ldots\alpha(n-1)\quad(\text{for } m\leqslant n),\quad\text{and}\quad \alpha(m,\omega):=\alpha(m)\alpha(m+1)\ldots$$

The logical connectives are written $\neg, \vee, \wedge, \rightarrow, \exists, \forall$. As a shorthand for the quantifiers "there exist infinitely many n" and "there are only finitely many n" we use "$\exists^\omega n$", respectively "$\exists^{<\omega} n$".

The following operations on sets of finite words are basic: for $W\subseteq A^*$ let

$$\text{pref}\, W:=\{u\in A^*\mid \exists v\, uv\in W\},$$
$$W^\omega:=\{\alpha\in A^\omega\mid \alpha=w_0w_1\ldots \text{ with } w_i\in W \text{ for } i\geqslant 0\},$$
$$\vec{W}:=\{\alpha\in A^\omega\mid \exists^\omega n\,\alpha(0,n)\in W\}.$$

Other notations for $\vec{W}$ found in the literature are $\lim W$ and W^δ. Finally, for an ω-sequence $\sigma=\sigma(0)\sigma(1)\ldots$ from S^ω, the "infinity set" of σ is

$$\text{In}(\sigma):=\{s\in S\mid \exists^\omega n\,\sigma(n)=s\}.$$

1. Büchi automata

Büchi automata are nondeterministic finite automata equipped with an acceptance condition that is appropriate for ω-words: an ω-word is accepted if the automaton can read it from left to right while assuming a sequence of states in which some final state ocurs infinitely often (*Büchi acceptance*).

DEFINITION. A *Büchi automaton* over the alphabet A is of the form $\mathcal{A}=(Q, q_0, \Delta, F)$ with finite state set Q, initial state $q_0\in Q$, transition relation $\Delta\subseteq Q\times A\times Q$, and a set $F\subseteq Q$ of final states. A *run* of $\mathcal{A}$ on an ω-word $\alpha=\alpha(0)\alpha(1)\ldots$ from A^ω is a sequence $\sigma=\sigma(0)\sigma(1)\ldots$ such that $\sigma(0)=q_0$ and $(\sigma(i), \alpha(i), \sigma(i+1))\in\Delta$ for $i\geqslant 0$; the run is called *successful* if $\text{In}(\sigma)\cap F\neq\emptyset$, i.e. some state of F occurs infinitely often in it. *$\mathcal{A}$ accepts* α if there is a successful run of $\mathcal{A}$ on α. Let

$$L(\mathcal{A})=\{\alpha\in A^\omega\mid \mathcal{A} \text{ accepts } \alpha\}$$

be the ω-language *recognized* by $\mathcal{A}$. If $L=L(\mathcal{A})$ for some Büchi automaton $\mathcal{A}$, L is said to be *Büchi recognizable*.

EXAMPLE. Consider the alphabet $A=\{a,b,c\}$. Define $L_1\subseteq A^\omega$ by

$\alpha\in L_1$ iff after any occurrence of letter a there is some occurrence of letter b in α.