Bogdanov's lemma proof

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1 Introduction

The goal of this document is to provide a formal graph theoretical proof of Bogdanov's lemma [1].

2 Bogdanov's lemma

Bogdanov's lemma states that a graph with only disjoint perfect matchings can have maximally three perfect matchings (PM) [4]. Bogdanov proved this using three steps:

1. Two disjoint PMs always form a Hamiltonian cycle.
2. A lower bound of the maximum amount of disjoint PMs can be easily found to be 2 (C_{2n}).
3. Edges can be added to try to increase the maximum amount of disjoint PMs. However, adding any new edges results in a third, non-disjoint PM.

These steps are also explained in a video by Mario Krenn [3].

3 Formal proof of Bogdanov's lemma

Definition 3.1 (Disjoint perfect matchings)**.** *Two perfect matchings are disjoint if they do not share any edges.*

Definition 3.2 (Exclusively disjoint perfect matching graph)**.** *An exclusively disjoint perfect matching graph is a (not necessarily simple) graph that contains only disjoint perfect matchings.*

Bogdanov's lemma stated formally (represented as a theorem):

Theorem 3.1 (Bogdanov's lemma)**.** *Given an exclusively disjoint perfect matching graph $G(V, E)$ with $|V| = 2n$. The maximum amount of perfect matchings G can have is*

$$c_{max}(2n) = \begin{cases} 3, & n = 2 \\ 2, & n > 2 \end{cases} \tag{1}$$

The first step of proving 3.1 is showing that two PMs of an exclusively disjoint perfect matching graph form a Hamiltonian cycle.

Lemma 3.2. *The union of any PMs M_1 and M_2 of an exclusively disjoint perfect matching graph G forms a Hamiltonian cycle.*

Proof. Let G' be the graph with edge set $M_1 \cup M_2$. Every vertex in $u \in G'$ is incident with an edge $uv \in M_1$ and $uw \in M_2$. This means that G' is two-regular, which is equivalent to saying that it is a disjoint union of cycles [2].

Any of the cycles in G' have even lengths of at least four. Because the cycles are disjoint, each of these cycles take the form of C_{2n} with $n \geq 2$. For m cycles in G' there exist 2^m PMs. Only two of these PMs are fully disjoint. This means that G can only be an exclusively disjoint perfect matching graph if $m = 1$. Because of this, G' can not contain multiple cycles and thus must form a Hamiltonian cycle. □

An exclusively disjoint perfect matching graph with two PMs constructed with as little edges as possible takes the shape of C_{2n}. Removing any single edge from such a graph also removes one of its PMs. The only way an exclusively disjoint perfect matching graph has three PMs is if it is at least 3-regular (i.e. every vertex has at least three neighbours). Therefore, the only way to increase the amount of PMs is to add edges.

Lemma 3.3. *Let G be an exclusively disjoint perfect matching graph of the form C_{2n} with $n > 2$. Let G' be another exclusively disjoint perfect matching graph constructed by adding any number of edges to G. G' can have maximally two PMs without losing its exclusively disjoint perfect matching property.*

Proof. G has an even number of vertices. This means an edge can be added that splits the cycle into two even or two odd length cycles.

Suppose an edge is added that splits the graph into two even cycles. Then start constructing a new PM, starting with the new edge. The two vertices that are connected by this edge do not need to be matched anymore. That leaves $2n - 2 = 2(n - 1)$ vertices that need to be matched, where $2(n - 1)$ is split up into two paths of even length. Both of these have a PM. Therefore, a new, third PM can be constructed. This new PM contains edges that either of the two pre-existing PMs also contain. This means that the new PM is not disjoint, and no edges can ever be added to G that split it into even cycles without removing the exclusively disjoint perfect matching property.

Conversely, a single edge that splits the graph into two odd cycles can never contribute to a PM, as the two cycles, with the newly matched pair of vertices removed, form two odd length paths, which do not have a PM individually. Therefore, more edges that split the graph into odd length cycles can be added. Suppose another odd length cycle subtending edge is added that neighbours and crosses the first edge. This splits up the original cycle graph into three parts. Because two of the vertices that are connected to the new edges are neighbours (and the edges cross each other), the individual cycles they formed are shortened by one. Therefore, two of the three parts are even length paths, which each contains a PM. The only vertices that are left to be accounted for are the vertices that form the third path created by the two edges. Consider the two odd length cycles that do not contain these vertices. The length of this last path is $2n - (\widetilde{n_1} + \widetilde{n_2})$, which is even or zero, so this path also contains a PM. Altogether, this newly formed PM is not disjoint with the original two PMs.

Finally, every cycle graph where every vertex is connected to a new such odd cycle subtending edge contains at least one of these vertex neighbour pairs that have crossing edges. This happens because the minimal odd cycle length is three, which leaves an unmatched vertex that needs to be matched. As its two neighbours are already matched, it needs to look for another vertex to match with, which requires their connecting edge to cross the first edge. $\square$

It has been shown that no new edges can be added to C_{2n} with $n > 2$ in order to get three PMs, without removing the exclusively disjoint perfect matching property. This proves that $c_{max}(2n) = 2$ for $n > 2$.

Furthermore, the only graph that is simple and 3-regular with four vertices is K_4, which can easily be seen to have three disjoint PMs.

References

[1] Ilya Bogdanov (https://mathoverflow.net/users/17581/ilya bogdanov). Graphs with only disjoint perfect matchings. MathOverflow. URL:https://mathoverflow.net/q/267013 (version: 2017-04-12).

[2] Math1000 (https://math.stackexchange.com/users/38584/math1000). Proof union of two disjoint perfect matchings in a bipartite graph is covering all vertices of g with disjoint cycles. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1728377 (version: 2016-04-05).

[3] Mario Krenn. An open conjecture in graph theory: Graph colorings and perfect matchings, 2020. Available online at `https://www.youtube.com/watch?v=07b1lIWX94k`.

[4] Mario Krenn. Quantum physics in lean, 2021. Available online at `https://mariokrenn.files.wordpress.com/2021/10/quantum_in_lean-1.pdf`.