1. Handling Obelix

I work with the free multiplicative group G on countably many generators, but an additive (abelian) free group also works. The functional equation for Obelix is

$$f(f^2(h)f(h)^{-1}) = hf(h)^{-1}.$$

Think of this as saying that if $(a,b),(b,c) \in f$, then we must have $(cb^{-1}, ab^{-1}) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

(1) E is finite.
(2) E is an injective function.
(3) $(1,1) \in E$.
(4) If $(a,b),(b,c) \in E$, then $(cb^{-1}, ab^{-1}) \in E$.
(5) If $(a,b) \in E$ and $a = b$, then $a = b = 1$.
(6) If $(a,b) \in E$ and $b \notin \mathrm{dom}(E)$, then $ab^{-1} \notin \mathrm{dom}(E) \cup \mathrm{im}(E)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a,b) \in E$ for some $b \in G$.

If $b \in \mathrm{dom}(E)$, then by condition (4) we are already done. So reduce to the case when $b \notin \mathrm{dom}(E)$. In particular, by (2) and (3) we know $a, b \neq 1$, and also by (6) we know that $ab^{-1} \notin \mathrm{dom}(E) \cup \mathrm{im}(E)$ (and is not 1).

Take c to be a generator of G not appearing in the reduced form of any entry in E, and set $E' := E \cup \{(b,c),(cb^{-1}, ab^{-1}\}$. Conditions (1), (3), and (5) are immediate. Condition (2) is also clear, where injectivity needs $ab^{-1} \notin \mathrm{im}(E)$. Condition (4) is also easy to check, using the fact that E is injective, $cb^{-1} \neq c$, and $ab^{-1} \notin \mathrm{dom}(E)$.

Finally, for condition (6), a finite check works. The main case is checking that $ca^{-1} \notin \mathrm{dom}(E') \cup \mathrm{im}(E')$. This is clear since $a \neq 1$ and $a \neq b$ (by condition (5) for E).

Case 2: Assume $a \notin \mathrm{dom}(E)$. If $(x,a) \in E$ for some (unique by (2)) $x \in G$, then applying Case 1 to x, we reduce to the case when $a \in \mathrm{dom}(E)$.

Thus, we may consider the case when $a \notin \mathrm{dom}(E) \cup \mathrm{im}(E)$. Fixing b to be a generator of G not appearing in the reduced forms for the entries in E, nor in a, then after passing to $E \cup \{(a,b)\} \in \mathscr{E}$ we again reduce to Case 1. □

The functional equation for Asterix is $f(f(f(h)h^{-1})h) = h$. Taking the seed

$$\{(1,1),(x_1,x_2),(x_2x_1^{-1},x_3),(x_3x_1,x_4)\} \in \mathscr{E}$$

works to contradict this equation.