On the formalisation of topological K-theory

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The industry of (co)homology

- General study of cohomology theories:
  - homological algebra
  - derived functors
  - simplicial objects
  - spectra / representability theorems
  - ...
- Studying a cohomology theory for its own sake:
  - ordinary cohomology
  - cobordism
  - sheaf cohomology
  - de Rham cohomology
  - ...
- Cohomology theory as a tool

I will argue that

Topological K-theory is:

- a powerful tool,
- ripe for construction within Mathlib,
- that would draw on several areas of the library.

Notation and boilerplate

```
import geometry.manifold.instances.sphere
import topology.vector_bundle.hom

noncomputable theory
open_locale manifold
variables {n : ℕ}

local notation `𝔼ⁿ⁺¹` := euclidean_space ℝ $ fin $ n+1
local notation `𝔼ⁿ` := euclidean_space ℝ $ fin n
local notation `𝕊ⁿ` := metric.sphere (0 : 𝔼ⁿ⁺¹) 1
local notation `T𝕊ⁿ` := (tangent_space (𝓡 n) : 𝕊ⁿ → Type*)
local notation `End_T𝕊ⁿ_at` :=
  bundle.continuous_linear_map (ring_hom.id ℝ) 𝔼ⁿ T𝕊ⁿ 𝔼ⁿ T𝕊ⁿ
local notation `End_T𝕊ⁿ` := bundle.total_space End_T𝕊ⁿ_at

instance : fact $ finite_dimensional.finrank ℝ 𝔼ⁿ⁺¹ = n + 1 :=
  ⟨finrank_euclidean_space_fin⟩
instance _ii (x : 𝕊ⁿ) : ring (End_T𝕊ⁿ_at x) :=
  continuous_linear_map.ring
```

K-theory as a tool

The following non-trivial result has an easy K-theoretic proof:

```
-- Borel, Serre (1953)
lemma sphere.eq_two_or_six_of_acs
  (J : 𝕊ⁿ → End_T𝕊ⁿ)
  (h₀ : bundle.total_space.proj ∘ J = id)
  (h₁ : continuous J)
  (h₂ : ∀ x, (J x).snd^2 = -1) :
  n = 2 ∨ n = 6 :=
sorry
```

K-theory proof of Borel-Serre

Recall that for any X we have the Chern character:

$$\begin{aligned}\text{ch} : K^*(X) &\to H^*(X, \mathbb{Q}) \\ [E] &\mapsto c_0 + c_1 + \frac{1}{2!}(c_1^2 - 2c_2) + \frac{1}{3!}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots\end{aligned}$$

where $c_i \in H^{2i}(X, \mathbb{Z})$ is the i^{th} Chern class of E.

Lemma (Karoubi (1964)). *For $X = S^n$:*

$$\text{range}(\text{ch}) = H^*(S^n, \mathbb{Z})$$

Corollary. *Let $E \to S^{2m}$ be a rank m complex vector bundle, and let $c_m(E) \in H^{2m}(S^{2m}, \mathbb{Z}) \simeq \mathbb{Z}$ be the top Chern class, then:*

$$(m-1)! \mid c_m(E)$$

In particular if $E \simeq TS^{2m}$ as real bundles, then since $\chi(S^{2m}) = 2$:

$$(m-1)! \mid 2,$$

and so $m \leq 3$.

Defining $K^0(X)$: preparation

```
import group_theory.monoid_localization

abbreviation K (A : Type*) [add_comm_monoid A] :=
add_localization (⊤ : add_submonoid A)

-- Sanity check:
noncomputable example : K ℕ ≃+ ℤ :=
add_localization.add_equiv_of_quotient
{ to_fun := coe,
  map_zero' := nat.cast_zero,
  map_add' := nat.cast_add,
  ... }
```

Homework:

- ‘Show that the localization at the top monoid is a group.’
- Equivalence with quotient of `add_monoid_algebra ℤ A` (or `free_abelian_group A`)

Defining $K^0(X)$: attempt 1

```
import topology.vector_bundle.constructions
import analysis.complex.basic

variables (B F : Type*) (E : B → Type*)
  [normed_add_comm_group F] [normed_space ℂ F]
  [topological_space B] [∀ x, topological_space (E x)]
  [∀ x, add_comm_monoid (E x)] [∀ x, module ℂ (E x)]
  [topological_space (bundle.total_space E)]
  [fiber_bundle F E]

#check vector_bundle ℂ F E -- Prop

example (F' : Type*) (E' : B → Type*) [...] :
  vector_bundle ℂ (F × F') (λ b, E b × E' b) :=
vector_bundle.prod ℂ F E F' E' -- Direct sum
```

Homework:

- Define morphisms and isomorphisms
- Construct the (exact) category of finite-rank vector bundles

Defining $K^0(X)$: attempt 2

How about finitely-generated projective modules over $C(X, \mathbb{C})$?

```
import topology.continuous_function.algebra
import topology.instances.complex
import algebra.category.Module.projective

open category_theory
variables (R X : Type*) [ring R] [topological_space X]

instance : add_comm_monoid (skeleton $ full_subcategory $
  λ P : Module R, module.finite R P ∧ projective P) :=
sorry

def ring.K := add_localization
  (⊤ : add_submonoid $ skeleton $ full_subcategory $
  λ P : Module R, module.finite R P ∧ projective P)

def topological_space.K [compact_space X] [t2_space X] :=
ring.K C(X, ℂ)
```

Defining $K^0(X)$: attempt 2

Homework

- Direct sum of finitely-generated modules is finitely-generated
- As above but for projectiveness
- Additive monoid structure on isomorphism classes (i.e., the `sorry`)
- Define via exact sequences rather than direct sums
- Serre-Swan theorem: very nice target!

Defining $K^0(X)$: attempt 3

```
import analysis.inner_product_space.l2_space
import topology.homotopy.basic

open_locale cardinal

variables {E : Type*}
  [normed_add_comm_group E] [normed_space ℂ E]

def is_fredholm (T : E →L[ℂ] E) : Prop :=
module.rank ℂ T.ker < ℵ₀ ∧ module.rank ℂ (E ⧸ T.range) < ℵ₀

variables (E)

def 𝓕 : submonoid (E →L[ℂ] E) :=
{ carrier := is_fredholm,
  one_mem' := sorry,
  mul_mem' := sorry, }

def topological_space.K (X : Type*) [topological_space X] :=
@quot C(X, 𝓕 ℓ²(ℕ, ℂ)) continuous_map.homotopic -- Success!
```

Natural filtration

If X is a finite CW complex we have the sequence of skeleta:

$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_d = X$$

K^0 is a contravariant functor, so associated to each inclusion $X_i \to X$ we have a map, and can thus consider:

$$K^0_{i+1}(X) = \ker(K^0(X) \to K^0(X_i))$$

This turns $K^0(X)$ into a filtered group (in fact ring) whose associated graded is the even (ordinary) cohomology.

Homework

- Define CW complexes. Connect with Mathlib's `sSet`.
- Define filtered rings (maybe using `ideal.filtration`)

What about K^n?

Recall suspension isomorphism from ordinary cohomology:

$$\tilde{H}^{n-1}(X) \simeq \tilde{H}^n(S^1 \wedge X)$$

Since $S^1 \wedge X$ is connected, this says $\tilde{H}^{-n}(X) = 0$ which is boring. However things are much more interesting for K^0! We can actually define:

$$\tilde{K}^{-n}(X) = \tilde{K}^0(S^n \wedge X)$$

There is then the amazing fact of **Bott periodicity**:

$$\tilde{K}^{-n-2}(X) \simeq \tilde{K}^{-n}(X)$$

So we get a filtered, $\mathbb{Z}/(2)$-graded (lambda) ring: $K^0(X) \oplus K^{-1}(X)$.

Homework

- Define pointed spaces
- Define wedge sum $X \vee Y$, smash product $X \wedge Y$
- Show $S^1 \wedge S^n \simeq S^{n+1}$

Cyclic exact sequence

Associated to $Y \subseteq X$, there is a natural exact sequence:

$$\begin{array}{ccccc} K^0(X,Y) & \longrightarrow & K^0(X) & \longrightarrow & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,Y) \end{array}$$

Does a cyclic exact sequence present any issues formally?

I think not! Maybe someone can comment?

Further connections

- Equivariant K-theory. Includes representation ring. Useful tool (e.g., for homogeneous spaces).

- Connection with bundles of Clifford modules (Atiyah-Bott-Shapiro).

- Many equivalent forms of Bott periodicity including the amazing fact $\Omega^2\mathcal{F} \simeq \mathcal{F}$.

- Atiyah-Hirzebruch spectral sequence.

- Adams operations.