A electrostatics of a point particle in 3d.

In this module we derive properties of the electrostatics of a point particle of charge `q` sitting in `3` d space.

i. Key results

- The electric potential is given by `electricPotential q ε r₀`.
- The electric field is given by `electricField q ε r₀`.
- Gauss's law is given in `gaussLaw`.
- Faraday's law is given in `faradaysLaw`.

ii. References

- The proof of Gauss' law in this module follows: https://math.stackexchange.com/questions/2409008/

A. Definitions

We start by stating the charge distribution, electric potential and electric field of the point particle. Later on in this module we will prove that these definitions are correct, by showing they satisfy the necessary physical properties.

We have the following definitions:

- The `chargeDistribution` is `q δ(r-r₀)`.
- The `electricPotential` is `(q/(4 * π * ε)) • ‖r - r₀‖⁻¹`.
- The `electricField` is `(q/(4 * π * ε)) • ‖r - r₀‖⁻¹ ^ 3 • (r - r₀)`.

```
def Electromagnetism.ThreeDimPointParticle.chargeDistribution
      (q : ℝ) (r₀ : Space) :
  ChargeDistribution
```
source

The charge distribution of a point particle of charge `q` in 3d space sitting at the `r₀`. In the physicists notation this corresponds to the 'function' `q δ(r-r₀)`.

▸ Equations
▸ Instances For

```
def Electromagnetism.ThreeDimPointParticle.electricPotential
      (q ε : ℝ) (r₀ : Space) :
  StaticElectricPotential
```
source

The electric potential of a point particle of charge `q` in 3d space sitting at the `r₀`. In physics notation this corresponds to the 'function' `(q/(4 * π * ε)) • ‖r - r₀‖⁻¹`. Here it is defined as the distribution corresponing to that function.

▸ Equations
▸ Instances For

```
def Electromagnetism.ThreeDimPointParticle.electricField
      (q ε : ℝ) (r₀ : Space) :
  StaticElectricField
```
source

The electric field of a point particle of charge `q` in 3d space sitting at `r₀`. In physics notation this corresponds to the 'function' `(q/(4 * π * ε)) • ‖r - r₀‖⁻¹ ^ 3 • (r - r₀)`. Here it is defined as the distribution corresponding to that function.

▸ Equations
▸ Instances For

B. Properties for `q = 0`

We first prove that the charge distribution, electric potential and electric field are all zero when the charge of the particle is zero.

```
theorem Electromagnetism.ThreeDimPointParticle.
      chargeDistribution_eq_zero_of_charge_eq_zero
      (r₀ : Space) :
  chargeDistribution 0 r₀ = 0
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
      electricPotential_eq_zero_of_charge_eq_zero
            (r₀ : Space) :
  electricPotential 0 ε r₀ = 0
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
      electricField_eq_zero_of_charge_eq_zero
            (r₀ : Space) :
  electricField 0 ε r₀ = 0
```
source

C. Translations

We now prove that the charge distribution, electric potential and electric field for the point particle at `r₀` is just the translation of the charge distribution, electric potential and electric field for the point particle located at `0`.

```
theorem Electromagnetism.ThreeDimPointParticle.chargeDistribution_eq_translateD  source
        (q : ℝ) (r₀ : Space) :
    chargeDistribution q r₀ = (Space.translateD r₀) (chargeDistribution q 0)
```

```
theorem Electromagnetism.ThreeDimPointParticle.electricPotential_eq_translateD  source
        (q ε : ℝ) (r₀ : Space) :
    electricPotential q ε r₀ = (Space.translateD r₀) (electricPotential q ε 0)
```

```
theorem Electromagnetism.ThreeDimPointParticle.electricField_eq_translateD  source
        (q ε : ℝ) (r₀ : Space) :
    electricField q ε r₀ = (Space.translateD r₀) (electricField q ε 0)
```

D. Proving the gradient of the potential is the electric field

We now prove that the electric field is equal to the negative gradient of the potential, i.e. `E = -∇φ`.

D.1. Reducing the problem to showing an integral is zero

Until the very end of this problem we will implicitly assume that `r₀ = 0`. We generalize at the end.

The first step of our proof is to show that `E = -∇φ` if for any Schwartz map `η` and direction `y` the integral `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ = 0` is equal to zero.

Recall that a 'Schwartz map' is a smooth function which, along with all it's derivatives, decays fast. It's presence here is because the electric field and potential are defined as distributions, and distributions are defined by how they act on Schwartz maps.

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
        gradD_electricPotential_eq_electricField_of_integral_eq_zero
        (q ε : ℝ)
        (h_integral :

         ∀ (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ) (y :
          EuclideanSpace ℝ (Fin 3)),

          ∫ (a : EuclideanSpace ℝ (Fin 3)), (((SchwartzMap.fderivCLM ℝ) η) a) y * ‖a
           ‖⁻¹ + η a * -inner ℝ ((‖a‖ ^ 3)⁻¹ • a) y
          =       0
         )
        :
    -Space.gradD (electricPotential q ε 0) = electricField q ε 0
```

The relation `E = -∇φ` holds for the point particle if the integral `∫ x, d_y η x * ‖x‖⁻¹ + η x * -⟪(‖x‖ ^ 3)⁻¹ • x, y⟫_ℝ = 0` is zero.

D.2. A smooth approximation to `‖r‖⁻¹`

Notice that in the integral `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ = 0` the integrand is has the structure of the total derivative of the function `η r * ‖r‖⁻¹` in the direction `y`, i.e. `d_y (η r * ‖r‖⁻¹)`.

However, this doesn't quite work because `‖r‖⁻¹` is not differentiable at `r = 0`. To get around this we define a sequence of functions, which for `n : ℕ` are given by `potentialLimitSeries n r = (‖r‖ ^ 2 + 1/(n + 1))^ (-1/2 : ℝ)`.

The overall aim will be to write `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ` as the limit of the integrals `∫ r, d_y η r * potentialLimitSeries n r + η r * d_y (potentialLimitSeries n) r y` as `n → ∞`, and then show that each of these integrals is zero because they are integrals of total derivatives of differentiable functions.

```
def Electromagnetism.ThreeDimPointParticle.potentialLimitSeries  source
        :
    ℕ → EuclideanSpace ℝ (Fin 3) → ℝ
```

A series of functions whose limit is the `‖x‖⁻¹` and for which each function is differentiable everywhere.

- Equations
- Instances For

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_eq  source
        (n : ℕ) :
    potentialLimitSeries n =
            fun (x : EuclideanSpace ℝ (Fin 3)) => (‖x‖ ^ 2 + 1 / (↑n + 1)) ^ (-1 / 2)
```

Part D.2.I.

The most important property of `potentialLimitSeries` is that it converges to `‖x‖⁻¹` as `n → ∞`. That is, it approximates `‖x‖⁻¹` arbitrarily closely for large enough `n`.

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_tendsto  source
        (x : EuclideanSpace ℝ (Fin 3)) (hx : x ≠ 0) :
    Filter.Tendsto (fun (n : ℕ) => potentialLimitSeries n x) Filter.atTop
     (nhds ‖x‖⁻¹)
```

Part D.2.II.

Unlike `‖r‖⁻¹`, importantly the functions `potentialLimitSeries n` are differentiable everywhere.

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_differentiable
        (n : ℕ) :
    Differentiable ℝ (potentialLimitSeries n)
```
source

Part D.2.III.

The derivative of `potentialLimitSeries n` in the direction `y` is given by `- (‖r‖^1 + 1/(1 + n))^(-3/2) * ⟪r, y⟫_ℝ`, or equivalently `- (potentialLimitSeries n r) ^ 3 * ⟪r, y⟫_ℝ`.

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_fderiv
        (x y : EuclideanSpace ℝ (Fin 3)) (n : ℕ) :
    (fderiv ℝ (potentialLimitSeries n) x) y = -((‖x‖ ^ 2 + (1 + ↑n)⁻¹) ^ (-1 / 2)) ^ 3
            * inner ℝ x y
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_fderiv_eq_potentialLimitseries_mul
        (x y : EuclideanSpace ℝ (Fin 3)) (n : ℕ) :
    (fderiv ℝ (potentialLimitSeries n) x) y = -potentialLimitSeries n x ^ 3 *
            inner ℝ x y
```
source

Part D.2.IV.

as `n → ∞` the limit of the derivative of `potentialLimitSeries n` in the direction `y` is `-⟪(‖x‖ ^ 3)⁻¹ • x, y⟫_ℝ`. This is exactly the derivative of `‖x‖⁻¹` in the direction `y`, when it exists (i.e. when `x ≠ 0`).

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_fderiv_tendsto
        (x y : EuclideanSpace ℝ (Fin 3)) (hx : x ≠ 0) :
    Filter.Tendsto (fun (n : ℕ) => (fderiv ℝ (potentialLimitSeries n) x) y)
     Filter.atTop
       (nhds (-inner ℝ ((‖x‖ ^ 3)⁻¹ • x) y))
```
source

Part D.2.V

Because we are integrating, we need to show some integrability and measurability properties of `potentialLimitSeries` and it's derivative.

We first show that they are almost everywhere strongly measurable.

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_aeStronglyMeasurable
        (n : ℕ) :
    MeasureTheory.AEStronglyMeasurable (potentialLimitSeries n)
     MeasureTheory.volume
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_fderiv_aeStronglyMeasurable
        (n : ℕ) (y : EuclideanSpace ℝ (Fin 3)) :
    MeasureTheory.AEStronglyMeasurable
     (fun (x : EuclideanSpace ℝ (Fin 3)) =>
      (fderiv ℝ (potentialLimitSeries n) x) y)
       MeasureTheory.volume
```
source

Part D.2.VI.

We now show that `potentialLimitSeries` satisfies the condition `IsDistBounded`. Along with the fact it is almost everywhere strongly measurable, this means it can be made into a tempered distribution, but for our purposes means that it is integrable when multiplied by a Schwartz map.

There are a number of precursory lemmas first.

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_eq_sqrt_inv
        (n : ℕ) :
    potentialLimitSeries n =
            fun (x : EuclideanSpace ℝ (Fin 3)) => √(‖x‖ ^ 2 + 1 / (↑n + 1))⁻¹
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_nonneg
        (n : ℕ) (x : EuclideanSpace ℝ (Fin 3)) :
    0 ≤ potentialLimitSeries n x
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
        potentialLimitSeries_bounded_neq_zero
        (n : ℕ) (x : EuclideanSpace ℝ (Fin 3)) (hx : x ≠ 0) :
    ‖potentialLimitSeries n x‖ ≤ ‖x‖⁻¹
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.potentialLimitSeries_bounded
        (n : ℕ) (x : EuclideanSpace ℝ (Fin 3)) :
    ‖potentialLimitSeries n x‖ ≤ ‖x‖⁻¹ + √(↑n + 1)
```
source

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeries_isDistBounded
      (n : ℕ) :
   Distribution.IsDistBounded (potentialLimitSeries n)
```

source

Part D.2.VII.

In a similar fashion, and for the same reason, we now show that the derivative of `potentialLimitSeries` satisfies the condition `IsDistBounded`.

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeries_fderiv_bounded
      (n : ℕ) (x y : EuclideanSpace ℝ (Fin 3)) :
   ‖(fderiv ℝ (potentialLimitSeries n) x) y‖ ≤ ‖x‖⁻¹ ^ 2 * ‖y‖
```

source

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeries_fderiv_isDistBounded
      (n : ℕ) (y : EuclideanSpace ℝ (Fin 3)) :
   Distribution.IsDistBounded
    fun (x : EuclideanSpace ℝ (Fin (Nat.succ 2))) =>
     (fderiv ℝ (potentialLimitSeries n) x) y
```

source

D.3. A series of integrals

We now show that the integral `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ` is the limit of the integrals `∫ r, d_y (η r * potentialLimitSeries n r)` as `n → ∞`.

Part D.3.I.

We first define a series of functions which are the integrands of `∫ r, d_y (η r * potentialLimitSeries n r)`. These functions are `potentialLimitSeriesFDerivSchwartz y η n r`.

```
def Electromagnetism.ThreeDimPointParticle.potentialLimitSeriesFDerivSchwartz
      (y : EuclideanSpace ℝ (Fin 3))
      (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ) (n : ℕ)
      (x : EuclideanSpace ℝ (Fin 3)) :
   ℝ
```

source

A series of functions of the form `fderiv ℝ (fun x => η x * potentialLimitSeries n x) x y`.

- Equations
- Instances For

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeriesFDerivSchwartz_eq
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      (n : ℕ) (x : EuclideanSpace ℝ (Fin 3)) :
   potentialLimitSeriesFDerivSchwartz y η n x = (fderiv ℝ (↑η) x) y *
         potentialLimitSeries n x + η x * (fderiv ℝ (potentialLimitSeries n) x) y
```

source

Part D.3.II.

We show that these integrands converge to the integrand of `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ` as `n → ∞`.

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeriesFDerivSchwartz_tendsto
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      :
   ∀ᵐ (a : EuclideanSpace ℝ (Fin 3)),
    Filter.Tendsto
     (fun (n : ℕ) => potentialLimitSeriesFDerivSchwartz y η n a) Filter.atTop

     (nhds ((fderiv ℝ (↑η) a) y * ‖a‖⁻¹ + η a * -inner ℝ ((‖a‖ ^ 3)⁻¹ • a) y))
```

source

Part D.3.III.

We use 'Lebesgue dominated convergence theorem' to show that the integrals `∫ r, d_y (η r * potentialLimitSeries n r)` converge to the integral `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ` as `n → ∞`.

This requires some measurability properties of `potentialLimitSeriesFDerivSchwartz` and uses the integrability properties of `potentialLimitSeries` and its derivative shown above.

```
theorem Electromagnetism.ThreeDimPointParticle.
      potentialLimitSeriesFDerivSchwartz_aeStronglyMeasurable
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      (n : ℕ) :
   MeasureTheory.AEStronglyMeasurable
    (fun (x : EuclideanSpace ℝ (Fin 3)) =>
     potentialLimitSeriesFDerivSchwartz y η n x)
      MeasureTheory.volume
```

source

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      potentialLimitSeriesFDerivSchwartz_integral_tendsto_eq_integral
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      :
   Filter.Tendsto
    (fun (n : ℕ) =>
     ∫ (x : EuclideanSpace ℝ (Fin 3)),
      potentialLimitSeriesFDerivSchwartz y η n x
    )
     Filter.atTop

    (nhds
     (∫ (x : EuclideanSpace ℝ (Fin 3)), (fderiv ℝ (↑η) x) y * ‖x‖⁻¹ + η x * -
      inner ℝ ((‖x‖ ^ 3)⁻¹ • x) y)
     )
```

D.4. The limit of the series of integrals is zero

Above we showed that the limit of the integrals `∫ r, d_y (η r * potentialLimitSeries n r)` as `n → ∞` is `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ`. We now show that this same limit is zero.

Part D.4.I.

The integral `∫ r, d_y (η r * potentialLimitSeries n r)` is zero for each `n : ℕ`. This follows because this integrand is the total derivative of a differentiable function.

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      potentialLimitSeriesFDerivSchwartz_integral_eq_zero
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      (n : ℕ) :
   ∫ (x : EuclideanSpace ℝ (Fin 3)),                                              = 0
    potentialLimitSeriesFDerivSchwartz y η n x
```

Part D.4.II.

From part D.4.I it follows that the limit of the integrals `∫ r, d_y (η r * potentialLimitSeries n r)` as `n → ∞` is zero, since each individual integral is zero.

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      potentialLimitSeriesFDerivSchwartz_integral_tendsto_eq_zero
      (y : EuclideanSpace ℝ (Fin 3)) (η : SchwartzMap (EuclideanSpace ℝ (Fin 3)) ℝ)
      :
   Filter.Tendsto
    (fun (n : ℕ) =>
     ∫ (x : EuclideanSpace ℝ (Fin 3)),
      potentialLimitSeriesFDerivSchwartz y η n x
    )
     Filter.atTop (nhds 0)
```

D.5. E = -∇ V for a particle at the origin

We now put everything together. In part D.1 we showed that `E = -∇ V` follows from the integral `∫ r, d_y η r * ‖r‖⁻¹ + η r * -⟪(‖r‖ ^ 3)⁻¹ • x, r⟫_ℝ = 0` for all Schwartz maps `η` and directions `y`. In part D.3 we showed that this integral is the limit of the integrals `∫ r, d_y (η r * potentialLimitSeries n r)` as `n → ∞`. In part D.4 we showed that this limit is zero, and therefore this integral itself must be zero.

It follows that `E = -∇ V` for a particle at the origin.

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      electricField_eq_neg_gradD_electricPotential_origin
      (q ε : ℝ) :
   electricField q ε 0 = -Space.gradD (electricPotential q ε 0)
```

D.6. E = -∇ V for a particle at r_0

The general case of a particle at `r₀` follows from the case of a particle at the origin by using that the gradient commutes with translation.

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      electricField_eq_neg_gradD_electricPotential
      (q ε : ℝ) (r₀ : EuclideanSpace ℝ (Fin 3)) :
   electricField q ε r₀ = -Space.gradD (electricPotential q ε r₀)
```

```
theorem Electromagnetism.ThreeDimPointParticle.                                  source
      electricField_eq_ofPotential_electricPotential
      (q ε : ℝ) (r₀ : EuclideanSpace ℝ (Fin 3)) :
   electricField q ε r₀ = StaticElectricField.ofPotential (electricPotential q ε r₀)
```

E. Faraday's law

Faraday's law, which says that `∇ × E = 0`, is an immediate consequence of the fact that `E = -∇ V`, because the curl of a gradient is always zero.

```
theorem Electromagnetism.ThreeDimPointParticle.faradaysLaw
      (q ε : ℝ) (r₀ : Space) :
    (electricField q ε r₀).FaradaysLaw
```
source

F. Gauss' law

We now prove Gauss' law for a point particle in 3-dimensions. Recall that Gauss' law states that the divergence of the electric field is equal to the charge density divided by the permittivity, i.e. `∇ • E = ρ/ε`.

In this case, this result is related to the sometimes confusing fact that `∇ • ((‖r‖⁻¹) ^ 3 • r) ∝ δ(r)`.

We first prove Gauss' law for a point particle at the origin, and then use translation to prove it for a point particle at `r₀`.

F.1. Gauss' law for a point particle at the origin

The proof of Gauss' law for a point particle at the origin follows the proof given here: https://math.stackexchange.com/questions/2409008/

We highlight the main steps of the proof here (the below comments also appear in-line within the proof):

- **Step 1**: `∇ · E = 1/ε ρ` if for all Schwartz maps `η`, `∇ · E η = (1/ε ρ) η`.
- **Step 2**: We focus on rewriting the LHS, by definition it is equal to `- ∫ d³r ⟪(q/(4 * π * ε)) • ‖r‖⁻¹ ^ 3 • r, (∇ η) r⟫`
- **Step 3**: We rearrange the integral to `- q/(4 * π * ε) * ∫ d³r ‖r‖⁻¹ ^ 2 * ⟪‖r‖⁻¹ • r), (∇ η) r⟫`
- **Step 4**: We use that `⟪‖r‖⁻¹ • r), (∇ η) r⟫ = (d(η (a • ‖r‖⁻¹ • r))/d a)_‖r‖` to rewrite the integral as `- q/(4 * π * ε) * ∫ d³r ‖r‖⁻¹ ^ 2 * (d(η (a • ‖r‖⁻¹ • r))/d a)_‖r‖`.
- **Step 5**: We move over to spherical coordinates rewriting `d³r` as `r² dr dn` where `dn` is the integral over the unit vectors `n`. In `d³r` the `r` is a vector whilst in `r² dr dn` the `r` is a scalar (the distance). `- q/(4 * π * ε) * ∫ dr² dr dn r⁻¹ ^ 2 * (d(η (a • n))/d a)_r`
- **Step 6**: The integral is rearanged to `- q/(4 * π * ε) * ∫ dn (∫_0^∞ r² dr r⁻¹ ^ 2 * (d(η (a • n))/d a)_r)`
- **Step 7**: The integral is further rearanged to `- q/(4 * π * ε) * ∫ dn (∫_0^∞ dr (d(η (a • n))/d a)_r)`
- **Step 8**: The inner integral `(∫_0^∞ dr (d(η (a • n))/d a)_r)` is an integral over a total derivative of a function which tends to zero at infinity, and so is equal to `-η 0`. Thus the integral is equal to `- q/(4 * π * ε) * ∫ dn (-η 0)`.
- **Step 9**: The integral `∫ dn` is equal to the surface area of the unit sphere, which is `4 * π`. And thus we get after some simplification `(q/ε) * η 0`.
- **Step 10**: This is manifestly equal to the right hand side `1/ε ρ η` since `ρ = q δ(r)`, thereby proving the result.

```
theorem Electromagnetism.ThreeDimPointParticle.gaussLaw_origin
      (q ε : ℝ) :
    (electricField q ε 0).GaussLaw ε (chargeDistribution q 0)
```
source

Guass' law for a point particle in 3-dimensions at the origin, that is this theorem states that the divergence of `(q/(4 * π * ε)) • ‖r‖⁻¹ ^ 3 • r` is equal to `q • δ(r)`.

F.2. Gauss' law for a point particle at `r₀`

We now show Gauss' law for a point particle at `r₀`. This follows from the case of a point particle at the origin by using that the divergence commutes with translation.

```
theorem Electromagnetism.ThreeDimPointParticle.gaussLaw
      (q ε : ℝ) (r₀ : EuclideanSpace ℝ (Fin 3)) :
    (electricField q ε r₀).GaussLaw ε (chargeDistribution q r₀)
```
source

G. Rotational invaraiance

We now prove the electric field, charge distribution and potential of a point particle are rotationally invariant.

This is yet to be done, and is a TODO item.

```
def Electromagnetism.ThreeDimPointParticle.electricField_rotationally_invariant
      :
    InformalLemma
```
source

The electrostatic field of a point particle is rotationally invariant.

▸ Equations
▸ Instances For