Fourier Analysis Handout

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1 LCA Groups

An LCA group is a locally compact abelian Hausdorff topological group. If G is an LCA group, then there exists a countably additive measure μ on the Borel σ-algebra of G satisfying the following properties:

- $\mu(S+g)=\mu(S)$ for all $g\in G$ and Borel $S\subseteq G$,
- $\mu(U)>0$ for all nonempty open $U\subseteq G$,
- $\mu(K)<\infty$ for all compact $K\subseteq G$,
- $\mu(S)=\inf\{\mu(U) : \text{open } U\supseteq S\}$ for all Borel $S\subseteq G$,
- $\mu(U)=\sup\{\mu(K) : \text{compact } K\subseteq U\}$ for all open $U\subseteq G$.

The measure μ is called a Haar measure, and is unique up to rescaling by positive real numbers.

2 Short Exact Sequences

A short exact sequence of LCA groups is a short exact sequence of continuous homomorphisms

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

such that the topologies on A and C are the subspace and quotient topologies from B. Then A must be a closed subgroup of B, so that $B/A\cong C$ is Hausdorff. Conversely, if A is a closed group of B, then the sequence

$$1 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 1$$

is a short exact sequence of LCA groups.

If $1\to A\to B\to C\to 1$ is a short exact sequence of LCA groups, then Haar measures da, db, dc are said to be compatible if they satisfy

$$\int_B f(b)\,db=\int_C\int_A f(ac)\,da\,dc$$

for all nice[1] functions $f\colon B\to\mathbb{C}$. The integral $\int_A f(ac)\,da$ is a function of $c\in B$ that is constant on each coset of A, and thus can be viewed as a function of $c\in C$. Fixing Haar measures on two of these three LCA groups uniquely determines a compatible Haar measure on the third.

[1] Nice means Schwartz-Bruhat. When $G=\mathbb{R}$, this condition states that all derivatives are rapidly decreasing.

3 The Fourier Transform

Let G be an LCA group. Let $\widehat{G}$ be the group of continuous homomorphisms $G \to \mathbb{T}$ under pointwise multiplication. Then $\widehat{G}$ is an LCA group under the compact-open topology, and is called the Pontryagin dual of G. The Pontryagin dual defines a contravariant functor from the category of LCA groups to itself. This functor is exact, is a duality of categories, and is its own inverse.

For a nice function $f\colon G \to \mathbb{C}$, the Fourier transform of F is the nice function $\widehat{f}\colon \widehat{G} \to \mathbb{C}$ defined by

$$\widehat{f}(\hat{g}) = \int_G f(g)\overline{\hat{g}(g)}\, dg.$$

The Fourier inversion formula states that there exists a unique dual measure $d\hat{g}$ on $\widehat{G}$ such that

$$f(g) = \int_{\widehat{G}} \widehat{f}(\hat{g})\hat{g}(g)\, d\hat{g}.$$

4 The Poisson Summation Formula

Let $1 \to A \to B \to C \to 1$ be a short exact sequence of LCA groups.

Lemma 1. *Let $f\colon B \to \mathbb{C}$ be a nice function. If the Haar measures da, db, dc are compatible, then the nice function $F\colon C \to \mathbb{C}$ defined by $F(c) = \int_A f(ac)\, da$ satisfies $\widehat{F}(\hat{c}) = \widehat{f}(\hat{c})$.*

Proof. Compatibility of the Haar measures da, db, dc gives

$$\widehat{F}(\hat{c}) = \int_C F(c)\overline{\hat{c}(c)}\, dc = \int_C \int_A f(ac)\overline{\hat{c}(c)}\, da\, dc = \int_C \int_A f(ac)\overline{\hat{c}(ac)}\, da\, dc = \int_B f(b)\overline{\hat{c}(b)}\, db = \widehat{f}(\hat{c}). \qquad \square$$

Theorem 2 (The Poisson Summation Formula)**.** *Let $f\colon B \to \mathbb{C}$ be a nice function. If the Haar measures da, db, dc are compatible, then $\int_A f(a)\, da = \int_{\widehat{C}} \widehat{f}(\hat{c})\, d\hat{c}$.*

Proof. The Fourier inversion theorem gives

$$\int_A f(a)\, da = F(1) = \int_{\widehat{C}} \widehat{F}(\hat{c})\hat{c}(1)\, dc = \int_{\widehat{C}} \widehat{f}(\hat{c})\, dc. \qquad \square$$

5 Volume One

Lemma 3. *Let G be a compact LCA group. Then the dual measure on $\widehat{G}$ is $\frac{1}{\mathrm{Vol}(G)}$ times counting measure.*

Proof. We will apply the Fourier inversion theorem to the constant function $1\colon G \to \mathbb{C}$. First note that

$$\widehat{1}(\hat{g}) = \int_G \overline{\hat{g}(g)}\,dg = \begin{cases} \mathrm{Vol}(G), & \text{if } \hat{g} = 1, \\ 0, & \text{if } \hat{g} \neq 1. \end{cases}$$

Then we have

$$1 = 1(g) = \int_{\widehat{G}} \widehat{1}(\hat{g})\hat{g}(g)\,d\hat{g} = \mathrm{Vol}(G) \cdot d\hat{g}(\{\hat{g}\}),$$

which shows that the dual measure on $\widehat{G}$ is $\frac{1}{\mathrm{Vol}(G)}$ times counting measure. $\square$

Let $1 \to A \to B \to C \to 1$ be a short exact sequence of LCA groups.

Lemma 4. *If the Haar measures da, db, dc are compatible, then the Haar measures $d\hat{a}, d\hat{b}, d\hat{c}$ are compatible.*

Proof. We have

$$\widehat{f}(\hat{b}) = \int_B f(b)\overline{\hat{b}(b)}\,db = \int_C \int_A f(ac)\overline{\hat{b}(ac)}\,da\,dc,$$

and

$$f(b) = \int_{\widehat{B}} \widehat{f}(\hat{b})\hat{b}(b)\,d\hat{b} = r\int_{\widehat{A}}\int_{\widehat{C}} \widehat{f}(\hat{a}\hat{c})\hat{a}(b)\hat{c}(b)\,d\hat{c}\,d\hat{a}$$

for some $r > 0$. Combining these gives

$$\begin{aligned}
f(1) &= r\int_{\widehat{A}}\int_{\widehat{C}}\int_C\int_A f(ac)\overline{(\hat{a}\hat{c})(ac)}\,da\,dc\,d\hat{c}\,d\hat{a} \\
&= r\int_{\widehat{A}}\int_{\widehat{C}}\int_C g_{\hat{a}}(c)\overline{\hat{c}(c)}\,dc\,d\hat{c}\,d\hat{a}, \text{ where } g_{\hat{a}}(c) = \int_A f(ac)\overline{\hat{a}(ac)}\,da \\
&= r\int_{\widehat{A}}\int_{\widehat{C}} \widehat{g_{\hat{a}}}(\hat{c})\,d\hat{c}\,d\hat{a} \\
&= r\int_{\widehat{A}} g_{\hat{a}}(1)\,d\hat{a} \\
&= r\int_{\widehat{A}}\int_A f(a)\overline{\hat{a}(a)}\,da\,d\hat{a} \\
&= r\int_{\widehat{A}} \widehat{f}(\hat{a})\,d\hat{a} \\
&= rf(1),
\end{aligned}$$

which forces $r = 1$. $\square$

Suppose that there are isomorphisms $A \cong \widehat{C}, B \cong \widehat{B}, C \cong \widehat{A}$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "\wr"'] & B \arrow[r] \arrow[d, "\wr"'] & C \arrow[r] \arrow[d, "\wr"'] & 1 \\
1 \arrow[r] & \widehat{C} \arrow[r] & \widehat{B} \arrow[r] & \widehat{A} \arrow[r] & 1
\end{tikzcd}$$

commute. We will assume that the composition $B \cong \widehat{B} \cong \widehat{\widehat{B}}$ agrees with the canonical isomorphism $B \cong \widehat{\widehat{B}}$.

Rescaling db by $r > 0$ will rescale $d\hat{b}$ by $\frac{1}{r}$. Then there exists a unique Haar measure db on B with the property that db matches $d\hat{b}$ across the isomorphism. We will give B this unique self-dual Haar measure.

Proposition 5. *If db matches $d\hat{b}$ across the isomorphism $B \cong \widehat{B}$, then da matches $d\hat{c}$ across the isomorphism $A \cong \widehat{C}$, and dc matches $d\hat{a}$ across the isomorphism $C \cong \widehat{A}$.*

Proof. When comparing $d\hat{c}$ and da across the isomorphism $A \cong \widehat{C}$, suppose that $d\hat{c}$ corresponds to $r\,da$.

Taking Pontryagin duals gives isomorphisms $\widehat{C} \cong \widehat{\widehat{A}}\,, \widehat{B} \cong \widehat{\widehat{B}}\,, \widehat{A} \cong \widehat{\widehat{C}}$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, "r", "\wr"'] & B \arrow[r] \arrow[d, "1", "\wr"'] & C \arrow[r] \arrow[d, "\frac{1}{r}", "\wr"'] & 1 \\
1 \arrow[r] & \widehat{C} \arrow[r] \arrow[d, "r", "\wr"'] & \widehat{B} \arrow[r] \arrow[d, "1", "\wr"'] & \widehat{A} \arrow[r] \arrow[d, "\frac{1}{r}", "\wr"'] & 1 \\
1 \arrow[r] & \widehat{\widehat{A}} \arrow[r] & \widehat{\widehat{B}} \arrow[r] & \widehat{\widehat{C}} \arrow[r] & 1
\end{tikzcd}$$

commute. Then we have

$$\int_A f(a)\,da = \int_{\widehat{C}} \widehat{f}(\hat{c})\,d\hat{c} = \int_{\widehat{\widehat{A}}} \widehat{\widehat{f}}\,(\hat{\hat{a}})\,d\hat{\hat{a}} = r^2 \int_A f(-a)\,da = r^2 \int_A f(a)\,da,$$

using the assumption that the composition $B \cong \widehat{B} \cong \widehat{\widehat{B}}$ agrees with the canonical isomorphism $B \cong \widehat{\widehat{B}}$. This equation forces $r = 1$, from which the proposition follows. $\square$

Theorem 6. *If db matches $d\hat{b}$ across the isomorphism $B \cong \widehat{B}$, and if A is discrete with counting measure, then C is compact with volume 1.*

Proof. The fact that C is compact follows from the general fact that Pontryagin duality interchanges discrete LCA groups and compact LCA groups. Since counting measure on A matches $\frac{1}{\mathrm{Vol}(C)}$ times counting measure on $\widehat{C}$, we must have $\mathrm{Vol}(C) = 1$. $\square$

Example 7. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ shows that $\mathrm{Vol}(\mathbb{R}/\mathbb{Z}) = 1$.

Example 8. The short exact sequence $0 \to K \to \mathbb{A} \to \mathbb{A}/K \to 0$ shows that $\mathrm{Vol}(\mathbb{A}/K) = 1$.