PART I. AUTOMATA ON INFINITE WORDS

Notation

Throughout this chapter A denotes a finite alphabet and A^*, respectively A^ω, denote the set of finite words, resp. the set of ω-sequences (or: *ω-words*) over A. An ω-word over A is written in the form $\alpha=\alpha(0)\alpha(1)\ldots$ with $\alpha(i)\in A$. Let $A^\infty=A^*\cup A^\omega$. Finite words are indicated by u, v, $w,\ldots$, the empty word by ε, and sets of finite words by U, V, $W,\ldots$ Letters α, $\beta,\ldots$ are used for ω-words and L, $L',\ldots$ for sets of ω-words (i.e., *ω-languages*). Notations for segments of ω-words are

$$\alpha(m,n):=\alpha(m)\ldots\alpha(n-1)\quad(\text{for } m\leqslant n),\quad\text{and}\quad \alpha(m,\omega):=\alpha(m)\alpha(m+1)\ldots$$

The logical connectives are written $\neg, \vee, \wedge, \rightarrow, \exists, \forall$. As a shorthand for the quantifiers "there exist infinitely many n" and "there are only finitely many n" we use "$\exists^\omega n$", respectively "$\exists^{<\omega} n$".

The following operations on sets of finite words are basic: for $W\subseteq A^*$ let

$$\text{pref } W:=\{u\in A^* \mid \exists v\, uv\in W\},$$
$$W^\omega:=\{\alpha\in A^\omega \mid \alpha=w_0w_1\ldots \text{ with } w_i\in W \text{ for } i\geqslant 0\},$$
$$\vec{W}:=\{\alpha\in A^\omega \mid \exists^\omega n\, \alpha(0,n)\in W\}.$$

Other notations for $\vec{W}$ found in the literature are $\lim W$ and W^δ. Finally, for an ω-sequence $\sigma=\sigma(0)\sigma(1)\ldots$ from S^ω, the "infinity set" of σ is

$$\text{In}(\sigma):=\{s\in S \mid \exists^\omega n\, \sigma(n)=s\}.$$

1. Büchi automata

Büchi automata are nondeterministic finite automata equipped with an acceptance condition that is appropriate for ω-words: an ω-word is accepted if the automaton can read it from left to right while assuming a sequence of states in which some final state ocurs infinitely often (*Büchi acceptance*).

DEFINITION. A *Büchi automaton* over the alphabet A is of the form $\mathcal{A}=(Q, q_0, \Delta, F)$ with finite state set Q, initial state $q_0\in Q$, transition relation $\Delta\subseteq Q\times A\times Q$, and a set $F\subseteq Q$ of final states. A *run* of $\mathcal{A}$ on an ω-word $\alpha=\alpha(0)\alpha(1)\ldots$ from A^ω is a sequence $\sigma=\sigma(0)\sigma(1)\ldots$ such that $\sigma(0)=q_0$ and $(\sigma(i), \alpha(i), \sigma(i+1))\in\Delta$ for $i\geqslant 0$; the run is called *successful* if $\text{In}(\sigma)\cap F\neq\emptyset$, i.e. some state of F occurs infinitely often in it. $\mathcal{A}$ *accepts* α if there is a successful run of $\mathcal{A}$ on α. Let

$$L(\mathcal{A})=\{\alpha\in A^\omega \mid \mathcal{A} \text{ accepts } \alpha\}$$

be the ω-language *recognized* by $\mathcal{A}$. If $L=L(\mathcal{A})$ for some Büchi automaton $\mathcal{A}$, L is said to be *Büchi recognizable*.

EXAMPLE. Consider the alphabet $A=\{a,b,c\}$. Define $L_1\subseteq A^\omega$ by

$\alpha\in L_1$ iff after any occurrence of letter a there is some occurrence of letter b in α.

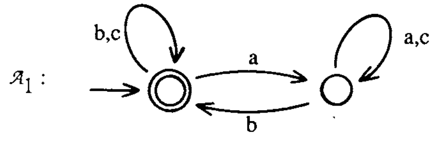

Fig. 1.

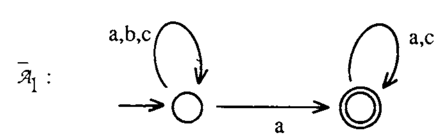

Fig. 2.

A Büchi automaton recognizing L_1 is (in state graph representation) shown in Fig. 1. The complement $A^{\omega} - L_1$ is recognized by the Büchi automaton in Fig. 2. Finally, the ω-language $L_2 \subseteq A^{\omega}$ with

$\alpha \in L_2$ iff between any two occurrences of letter a in α there is an even number of letters b, c

is recognized by the automaton shown in Fig. 3.

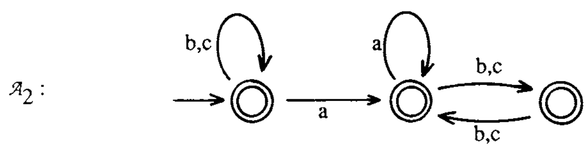

Fig. 3.

For a closer analysis of Büchi recognizable ω-languages we use the following notations, given some fixed Büchi automaton $\mathcal{A} = (Q, q_0, \Delta, F)$: if $w = a_0 \ldots a_{n-1}$ is a finite word over A, write $s \underset{w}{\rightarrow} s'$ if there is a state sequence $s_0, \ldots, s_n$ such that $s_0 = s$, $(s_i, a_i, s_{i+1}) \in \Delta$ for $i < n$, and $s_n = s'$. Let

$$W_{ss'} = \{w \in A^* \mid s \underset{w}{\rightarrow} s'\}.$$

Each of the finitely many languages $W_{ss'}$ is regular. By definition of Büchi acceptance, the ω-language recognized by $\mathcal{A}$ is

$$L(\mathcal{A}) = \bigcup_{s \in F} W_{q_0 s} . (W_{ss})^{\omega}. \tag{1.1}$$

1.1. THEOREM (Büchi [9]). *An ω-language $L \subseteq A^{\omega}$ is Büchi recognizable iff L is a finite union of sets $U . V^{\omega}$ where U, $V \subseteq A^*$ are regular sets of finite words (and where, moreover, one may assume $V . V \subseteq V$).*

In the proof, the direction from left to right is clear from equation (1.1); note that $W_{ss} . W_{ss} \subseteq W_{ss}$. For the converse, we verify the following closure properties of Büchi recognizable sets.

1.2. LEMMA. (a) *If $V \subseteq A^*$ is regular, then V^{ω} is Büchi recognizable.*

(b) *If $U \subseteq A^*$ is regular and $L \subseteq A^{\omega}$ is Büchi recognizable, then $U . L$ is Büchi recognizable.*

(c) *If* L_1, $L_2 \subseteq A^\omega$ *are Büchi recognizable, then* $L_1 \cup L_2$ *and* $L_1 \cap L_2$ *are Büchi recognizable.*

PROOF. (*a*) Since $V^\omega = (V - \{\varepsilon\})^\omega$, assume that V does not contain the empty word; further suppose that there is no transition into the initial state q_0 of the finite automaton which recognizes V. A Büchi automaton $\mathcal{A}'$ recognizing V^ω is obtained from the given automaton $\mathcal{A}$ by adding a transition (s, a, q_0) for any transition (s, a, s') with $s' \in F$, and by declaring q_0 as single final state of $\mathcal{A}'$.

The claims in (b) and (c) concerning concatenation and union are proved in the same way as for regular sets of finite words. For later use we show closure of Büchi recognizable sets under intersection. Suppose L_1 is recognized by $\mathcal{A}_1 = (Q_1, q_1, \Delta_1, F_1)$ and L_2 by $\mathcal{A}_2 = (Q_2, q_2, \Delta_2, F_2)$. A Büchi automaton recognizing $L_1 \cap L_2$ is of the form $\mathcal{A} = (Q_1 \times Q_2 \times \{0, 1, 2\}, (q_1, q_2, 0), \Delta, F)$, where the transition relation Δ copies Δ_1 and Δ_2 in the first two components of states, and changes the third component from 0 to 1 when an F_1-state occurs in the first component, from 1 to 2 when subsequently an F_2-state occurs in the second component and back to 0 immediately afterwards. Then 2 occurs infinitely often as third component in a run iff some F_1-state and some F_2-state occur infinitely often in the first two components. Hence with $F := Q_1 \times Q_2 \times \{2\}$ we obtain a Büchi automaton as desired. □

A representation of an ω-language in the form $L = \bigcup_{i=1}^{n} U_i . V_i^\omega$, where the U_i, V_i are given by regular expressions, is called an *ω-regular expression*. Since the constructions in Lemma 1.2 are effective, the conversion of ω-regular expressions into Büchi automata and vice versa can be carried out effectively. Hence, Büchi recognizable ω-languages are called *regular ω-languages*; other terms used in the literature are *ω-regular*, *rational*, *ω-rational*.

The formalism of ω-regular expressions can also be applied to *relations* over finite words instead of languages. In the case of binary relations, say for $S_1, S_2 \subseteq A^* \times B^*$, one defines $S_1 . S_2^\omega \subseteq A^\omega \times B^\omega$ by applying the operations of product and ω-power componentwise. The *rational ω-relations* are obtained as finite unions of relations $S_1 . S_2^\omega$ where S_1 and S_2 are rational relations over finite words (i.e., generated via usual regular expressions, cf. [5] or [27]). The rational ω-relations are characterized by a multitape version of Büchi automata with one reading head for each tape (Gire and Nivat [38]). However, as for rational relations over finite words, certain closure and decidability results cannot be transferred from regular ω-languages to the case of ω-relations (for instance, Theorems 2.1 and 2.3 below). In the sequel we restrict ourselves to ω-languages.

As is clear from equation (1.1), a Büchi automaton $\mathcal{A}$ accepts some ω-word iff $\mathcal{A}$ reaches some final state (say via the word u) which can then be revisited by a loop (say via the word v). The existence of a reachable final state which is located in a loop of $\mathcal{A}$ can be checked by an effective procedure. Hence, we obtain the following theorem.

1.3. THEOREM. (a) *Any nonempty regular ω-language contains an ultimately periodic ω-word* (*i.e., an ω-word of the form uvvv* ...).

(b) *The emptiness problem for Büchi automata is decidable.*

Vardi and Wolper [131] show that the nonemptiness-problem for Büchi automata ("$L(\mathscr{A}) \neq \emptyset$?") is logspace complete for NLOGSPACE, and Sistla, Vardi and Wolper [107] prove that the nonuniversality problem for Büchi automata ("$L(\mathscr{A}) \neq A^\omega$?") is logspace complete for PSPACE.

2. Congruences and complementation

Closure of Büchi recognizable ω-languages under complement is nontrivial and involves an interesting combinatorial argument. As will be seen in Section 4, it is not possible to work with a reduction to deterministic Büchi automata.

2.1. THEOREM (Büchi [9]). *If $L \subseteq A^\omega$ is Büchi recognizable, so is $A^\omega - L$. Moreover, from a Büchi automaton recognizing L one can construct one recognizing $A^\omega - L$.*

For the proof we shall represent both L and $A^\omega - L$ as finite unions of sets $U.V^\omega$ where U and V are regular sets of a special kind, namely classes of a certain congruence relation over A^* of finite index. (A congruence is here an equivalence relation compatible with concatenation.) Let $L = L(\mathscr{A})$ where $\mathscr{A} = (Q, q_0, \Delta, F)$ is a Büchi automaton. Write $s \xrightarrow[w]{F} s'$ if there is a run of $\mathscr{A}$ on w from state s to state s' such that at least one of the states in the run (including s and s') belongs to F. The set

$$W^F_{ss'} := \{w \in A^* \mid s \xrightarrow[w]{F} s'\}$$

is regular. Now define the equivalence relation $\sim_{\mathscr{A}}$ over A^* as follows:

$$u \sim_{\mathscr{A}} v \text{ iff } \forall s, s' \in Q\, (s \xrightarrow[u]{} s' \Leftrightarrow s \xrightarrow[v]{} s' \text{ and } s \xrightarrow[u]{F} s' \Leftrightarrow s \xrightarrow[v]{F} s').$$

The relation $\sim_{\mathscr{A}}$ is a congruence over A^*, which is of finite index by finiteness of Q. Hence each $\sim_{\mathscr{A}}$-class is regular. (The $\sim_{\mathscr{A}}$-class $[w]$ containing the word w is the intersection of the sets $W_{ss'}$ and $W^F_{ss'}$ and $A^* - W_{ss'}$ and $A^* - W^F_{ss'}$ containing w.)

A representation of $L(\mathscr{A})$ and $A^\omega - L(\mathscr{A})$ in terms of the $\sim_{\mathscr{A}}$-classes will be provided by the following lemma:

2.2. LEMMA. (a) *Let $\mathscr{A}$ be a Büchi automaton. For any $\sim_{\mathscr{A}}$-classes U, V: If $U.V^\omega \cap L(\mathscr{A}) \neq \emptyset$, then $U.V^\omega \subseteq L(\mathscr{A})$. (Hence: if $U.V^\omega \cap (A^\omega - L(\mathscr{A})) \neq \emptyset$, then $U.V^\omega \subseteq A^\omega - L(\mathscr{A})$.)*

(b) *Let $\sim$ be a congruence over A^* of finite index. For any ω-word $\alpha \in A^\omega$ there are $\sim$-classes U, V (even with $V.V \subseteq V$) such that $\alpha \in U.V^\omega$.*

Before the proof let us apply the lemma. Part (a) states a "saturation" property of $\sim_{\mathscr{A}}$ with respect to $L(\mathscr{A})$ and $A^\omega - L(\mathscr{A})$. By definition, a congruence $\sim$ over A^* *saturates* an ω-language $L \subseteq A^\omega$ if

$$U.V^\omega \cap L \neq \emptyset \text{ implies } U.V^\omega \subseteq L \quad \text{for all } \sim\text{-classes } U, V.$$

If $\sim$ saturates L and also is of finite index, then

$$L = \bigcup \{U.V^\omega \mid U, V \ \sim\text{-classes}, U.V^\omega \cap L \neq \emptyset\};$$

the inclusion "$\supseteq$" holds by saturation, and "$\subseteq$" follows from Lemma 2.2(b). Moreover, by the assumption that $\sim$ has finite index the $\sim$-classes are regular and the union is finite; so L is a regular ω-language. For the congruence $\sim_{\mathscr{A}}$, which saturates $A^{\omega} - L(\mathscr{A})$ by Lemma 2.2(a) and which is of finite index, we obtain that $A^{\omega} - L(\mathscr{A})$ is regular. Note that emptiness of $U.V^{\omega} \cap L(\mathscr{A})$ (and hence also nonemptiness of $U.V^{\omega} \cap (A^{\omega} - L(\mathscr{A}))$ can be decided effectively by Lemma 1.2(c) and Theorem 1.3. Hence, using Theorem 1.1, a Büchi automaton recognizing $A^{\omega} - L(\mathscr{A})$ can be constructed effectively from the given automaton $\mathscr{A}$. So Lemma 2.2 suffices to prove Theorem 2.1.

PROOF OF LEMMA 2.2. Let $\mathscr{A} = (Q, q_0, \Delta, F)$. Suppose $\alpha = uv_1v_2\ldots$ where $u \in U$ and $v_i \in V$ for $i > 0$, and assume further that there is a successful run of $\mathscr{A}$ on α. From this run we obtain states $s_1, s_2, \ldots$ such that

$$q_0 \underset{u}{\rightarrow} s_1 \underset{v_1}{\rightarrow} s_2 \underset{v_2}{\rightarrow} s_3 \rightarrow \cdots$$

where we even have

$$s_i \overset{F}{\underset{v_i}{\rightarrow}} s_{i+1} \text{ for infinitely many } i.$$

Let $\beta \in U.V^{\omega}$ be arbitrary. We show that $\beta \in L(\mathscr{A})$. We have $\beta = u'v'_1v_2\ldots$ where $u' \in U$ and $v'_i \in V$ for $i > 0$. Since U, V are $\sim_{\mathscr{A}}$-classes and hence $u \sim_{\mathscr{A}} u'$, $v_i \sim_{\mathscr{A}} v'_i$, we obtain

$$q_0 \underset{u}{\rightarrow} s_1 \underset{v_1}{\rightarrow} s_2 \underset{v_2}{\rightarrow} s_3 \rightarrow \cdots$$

and

$$s_i \overset{F}{\underset{v_i}{\rightarrow}} s_{i+1} \text{ for infinitely many } i.$$

This yields a run of $\mathscr{A}$ on β in which some F-state occurs infinitely often. Hence, $\beta \in L(\mathscr{A})$.

(b) Let $\sim$ be a congruence of finite index over A^*. Given $\alpha \in A^{\omega}$, two positions k, k' are said to *merge at position m* (where $m > k, k'$) if $\alpha(k, m) \sim \alpha(k', m)$. In this case write $k \cong_{\alpha} k'(m)$. Note that then also $k \cong_{\alpha} k'(m')$ for any $m' > m$ (because $\alpha(k, m) \sim \alpha(k', m)$ implies $\alpha(k, m)\alpha(m, m') \sim \alpha(k', m)\alpha(m, m')$). Write $k \cong_{\alpha} k'$ if $k \cong_{\alpha} k'(m)$ for some m. The relation $\cong_{\alpha}$ is an equivalence relation of finite index over ω (because $\sim$ is of finite index). Hence there is an infinite sequence $k_0, k_1, \ldots$ of positions which all belong to the same $\cong_{\alpha}$-class. By passing to a subsequence (if necessary), we can assume $k_0 > 0$ and that for $i > 0$ the segments $\alpha(k_0, k_i)$ all belong to the same $\sim$-class V. Let U be the $\sim$-class of $\alpha(0, k_0)$. We obtain

$$\exists k_0\, (\alpha(0, k_0) \in U \wedge \exists^{\omega} k\, (\alpha(k_0, k) \in V \wedge \exists m\, k_0 \cong_{\alpha} k\, (m))). \tag{2.1}$$

We shall show that (2.1) implies $\alpha \in U.V^{\omega}$ and $V.V \subseteq V$ (which completes the proof of (b)). Suppose that k_0 and a sequence $k_1, k_2, \ldots$ are given as guaranteed by (2.1). Again by passing to an infinite subsequence, we may assume that for all $i \geqslant 0$, the positions $k_0, \ldots, k_i$ merge at some $m < k_{i+1}$ and hence at k_{i+1}. We show $\alpha(k_i, k_{i+1}) \in V$ for $i \geqslant 0$. From (2.1) it is clear that $\alpha(k_0, k_1) \in V$. By induction assume that $\alpha(k_j, k_{j+1}) \in V$ for $j < i$. We know $\alpha(k_0, k_{i+1}) \in V$ and that k_0, k_i merge at k_{i+1}. Thus $\alpha(k_i, k_{i+1}) \in V$ and hence $\alpha \in U.V$.

Finally, in order to verify the claim $V.V \subseteq V$, it suffices to show $V.V \cap V \neq \emptyset$ (since V is a class of a congruence). But this is clear since $\alpha(k_0, k_i)$, $\alpha(k_i, k_{i+1})$ and $\alpha(k_0, k_{i+1})$ belong to V for any $i > 0$. □

The use of the merging relation $\cong_\alpha$ in the preceding proof can be avoided if Ramsey's Theorem is invoked (as done in the original proof by Büchi [9]): One notes that $\sim$ induces a finite partition of the set $\{(i,j) \mid i<j\}$ by defining that (i,j) and (i',j') belong to the same class iff $\alpha(i,j) \sim \alpha(i',j')$. Now Ramsey's Theorem (in the version for countable sets) states that there is an infinite "homogeneous" set, i.e., a set $\{i_0, i_1, \ldots\}$ such that all segments $\alpha(i_k, i_l)$ with $k<l$ are in one $\sim$-class, in particular all $\alpha(i_k, i_{k+1})$ are in this class. Define V to be this $\sim$-class and let U be the $\sim$-class of $\alpha(0, i_0)$. Then $\alpha \in U.V^\omega$. We gave the above self-contained proof because condition (2.1) will be used again in Section 4.

By Lemma 1.2 and Theorem 2.1, the regular ω-languages are effectively closed under Boolean operations. Hence the inclusion test and the equivalence test for Büchi automata reduce to the emptiness test (using $L_1 \subseteq L_2$ iff $L_1 \cap \sim L_2 = \emptyset$).

2.3. THEOREM. *The inclusion problem and the equivalence problem for Büchi automata are decidable.*

Let us consider the complexity of the complementation process and the equivalence test. Given a Büchi automaton with n states, there are n^2 different pairs (s, s') and hence $O(2^{2n^2})$ different $\sim_{\mathscr{A}}$-classes. This leads to a size bound of $O(2^{4n^2})$ states for the complement automaton [83, 107]. An improved and essentially optimal bound of $2^{O(n \log n)}$ is given in [99]. In [107] it is shown that the equivalence problem for Büchi automata is logspace complete for PSPACE.

The equivalence problem has also been studied in terms of equations between ω-regular expressions, building on work of Salomaa for classical regular expressions. A sound and complete axiom system for deriving these equations is given in [133]; for further references see [110].

Lemma 2.2 above not only shows complementation for regular ω-languages but also serves as a starting point for an investigation of these ω-languages in terms of finite semigroups. Recall that a language $W \subseteq A^*$ is regular iff there is a finite monoid M and a monoid homomorphism $f: A^* \to M$ such that W is a union of sets $f^{-1}(m)$ where $m \in M$. Since $A^*/\sim_{\mathscr{A}}$ is a finite monoid (for any Büchi automaton $\mathscr{A}$), we obtain, from Theorem 1.1 and Lemma 2.2, the following theorem.

2.4. THEOREM. *An ω-language $L \subseteq A^\omega$ is regular iff there is a finite monoid M and a monoid homomorphism $f: A^* \to M$ such that L is a union of sets $f^{-1}(m).(f^{-1}(e))^\omega$ with $m, e \in M$ and where e can be assumed to be idempotent (i.e., satisfying $e.e = e$).*

As will be shown in Theorem 2.6 below, there is a canonical minimal monoid with this property. In other words, there is a coarsest congruence $\approx_L$ over A^* which saturates L. We introduce $\approx_L$ here together with another natural congruence associated with an ω-language L. Given $L \subseteq A^\omega$, define for $u, v \in A^*$

$$u \sim_L v \quad \text{iff} \quad \forall \alpha \in A^\omega \, (u\alpha \in L \Leftrightarrow v\alpha \in L)$$

(a right congruence, cf. [124, 108]);

$$u \approx_L v \quad \text{iff} \quad \forall x, y, z \in A^* \, (xuyz^\omega \in L \Leftrightarrow xvyz^\omega \in L$$
$$\text{and } x(yuz)^\omega \in L \Leftrightarrow x(yvz)^\omega \in L)$$

(cf. [2]). The congruence $\approx_L$ regards two finite words as equivalent iff they cannot be distinguished by L as corresponding segments of ultimately periodic ω-words. Regularity of L implies that $\sim_L$ and $\approx_L$ are of finite index (since they are refined by the finite congruence $\sim_{\mathscr{A}}$ if $\mathscr{A}$ is a Büchi automaton that recognizes L). We note that the converse fails.

2.5. REMARK (Trakhtenbrot [124]). *There are nonregular sets $L \subseteq A^\omega$ such that $\sim_L$ and $\approx_L$ are of finite index.*

PROOF. For given $\beta \in A^\omega$, let $L(\beta)$ contain all ω-words that have a common suffix with β. Then any two words u, v are $\sim_{L(\beta)}$-equivalent since, for two ω-words $u\alpha$, $v\alpha$, membership in $L(\beta)$ does not depend on u, v. So there is only one $\sim_{L(\beta)}$-class. If we choose β to be not ultimately periodic, $L(\beta)$ is not regular (by Theorem 1.3). Furthermore, in this latter case also $\approx_{L(\beta)}$ has only one congruence class. □

We now show the mentioned maximality property of $\approx_L$.

2.6. THEOREM (Arnold [2]). *An ω-language L is regular iff $\approx_L$ is of finite index and saturates L; moreover, $\approx_L$ is the coarsest congruence saturating L.*

PROOF. If $\approx_L$ is of finite index and saturates L, then $L = \bigcup\{U . V^\omega \mid U, V$ are $\approx_L$-classes, $U . V^\omega \cap L \neq \emptyset\}$ and L hence is regular (see the remark following Lemma 2.2). Conversely, suppose L is regular; then (as seen before Remark 2.5) $\approx_L$ is of finite index. We show that $\approx_L$ saturates L, i.e. $U . V^\omega \cap L \neq \emptyset$ implies $U . V^\omega \subseteq L$ for any $\approx_L$-classes U, V. Since $U . V^\omega \cap L$ is regular, we can assume that there is an ultimately periodic ω-word xy^ω in $U . V^\omega \cap L$. In a decomposition of xy^ω into a U-segment and a sequence of V-segments, we find two V-segments which start after the same prefix y_1 of period y; so we obtain $w := xy^m y_1 \in U . V^r$ and $z := y_2 y^n y_1 \in V^s$ for some m, n, r, s and $y_1 y_2 = y$, so that $xy^\omega = wz^\omega$. Denote by $[w]$ and $[z]$ the $\approx_L$-classes of w and z. Since $[w] \cap U . V^r \neq \emptyset$ we have $U . V^r \subseteq [w]$; similarly, $V^s \subseteq [z]$, and hence $U . V^\omega \subseteq [w] . [z]^\omega$. It remains to prove $[w] . [z]^\omega \subseteq L$. For a contradiction, assume there is $\alpha \in [w] . [z]^\omega - L$, say $\alpha = w_0 z_1 z_2 \ldots$ where $w_0 \approx_L w$, $z_i \approx_L z$. Since α may be assumed again to be ultimately periodic, we obtain p, q with

$$\alpha = w_0 z_1 \ldots z_p (z_{p+1} \ldots z_{p+q})^\omega.$$

But then, from $wz^\omega = xy^\omega \in L$, we know $wz^p (z^q)^\omega \in L$, so

$$w_0 z_1 \ldots z_p (z_{p+1} \ldots z_{p+q})^\omega \in L$$

by definition of $\approx_L$ and thus $\alpha \in L$, a contradiction.

It remains to show that $\approx_L$ is the coarsest among the congruences $\sim$ saturating L. So assume $\sim$ is such a congruence and suppose $u \sim v$ (or: $\langle u \rangle = \langle v \rangle$ for the $\sim$-classes of

u and v). We verify $u \approx_L v$. We have $xuyz^\omega \in L$ iff $\langle xuy\rangle\langle z\rangle^\omega \subseteq L$ (since $\sim$ saturates L) iff $\langle xvy\rangle\langle z\rangle^\omega \subseteq L$ (since $u \sim v$) iff $xvyz^\omega \in L$. Similarly, one obtains $x(yuz)^\omega \in L$ iff $x(yvz)^\omega \in L$; thus $u \approx_L v$. □

The preceding result justifies calling $A^*/\approx_L$ the *syntactic monoid of* L, with concatenation of classes as the product. It allows us to classify the regular ω-languages by reference to selected varieties of monoids, extending the classification theory for regular sets of finite words (cf. [87]). Examples will be mentioned in Section 6.

3. The sequential calculus

One motivation for considering automata on infinite sequences was the analysis of the "sequential calculus", a system of monadic second-order logic for the formalization of properties of sequences. Büchi [9] showed the surprising fact that any condition on sequences that is written in this calculus can be reformulated as a statement about acceptance of sequences by an automaton.

For questions of logical definability, an ω-word $\alpha \in A^\omega$ is represented as a model-theoretic structure of the form $\underline{\alpha} = (\omega, 0, +1, <, (Q_a)_{a\in A})$, where $(\omega, 0, +1, <)$ is the structure of the natural numbers with zero, successor function, and the usual ordering, and where $Q_a = \{i \in \omega \mid \alpha(i) = a\}$ (for $a \in A$). The corresponding first-order language contains variables $x, y, \ldots$ for natural numbers, i.e. for the positions in ω-words. Typical atomic formulas are "$x+1<y$" ("the position following x comes before y") or "$x \in Q_a$" ("position x carries letter a"). In this framework, the example set $L_1 \subseteq \{a, b, c\}^\omega$ of Section 1 (containing the ω-words where after any letter a there is eventually a letter b) can be defined by the sentence

p. 11

We shall also allow variables $X, Y, \ldots$ for *sets* of natural numbers and quantifiers ranging over them. For example, they occur in a definition of the ω-language L_2 of Section 1 (containing the ω-words where between any two succeeding occurrences of letter a there is an even number of letters b, c):

$$\varphi_2:\quad \forall x\,\forall y\,(x \in Q_a \wedge y \in Q_a \wedge x<y \wedge \neg\exists z\,(x<z \wedge z<y \wedge z \in Q_a) \\ \rightarrow \exists X\,(x \in X \wedge \forall z\,(z \in X \leftrightarrow \neg(z+1 \in X)) \wedge \neg(y \in X))).$$

Note that the set quantifier postulates a set containing every second position starting with position x; this ensures that the number of letters between positions x and y is even. The sequential calculus consists of all the conditions on ω-words which can be written in this logical language.

One also calls this framework S1S for "second-order theory of *one* successor". (Below, in Theorem 3.1, it will be seen that $<$ is second-order definable in terms of successor and hence inessential). It is a system of *monadic second-order logic*, due to the quantification over sets, which are unary relations and hence "monadic second-order objects".