External Aggregation of Mathematics

May 19, 2020

2020 session program The aggregation test program is not written as a lesson plan. It describes a set of knowledge that the candidate must master and know how to demonstrate. It includes redundancies when concepts come naturally from different points of view. The program evokes examples sometimes, which are given for information only and can be replaced by others that would also be relevant.

In the sections 1 to 5 which follow, all the fields (denoted K in general) are assumed to be commutative.

1 Linear Algebra

1.1 Vector spaces

- Vector spaces, linear functions, products of vector spaces, subspaces, images and kernels of linear function, quotient spaces, sums of subspaces, direct sums. Free families, generative families, basis, linear vector space endomorphism algebras, linear group GL(E)
- Stable subspaces of an endomorphism. Proper value, proper vectors, proper subspaces
- Linear representation of a group. Irreducibility. In finite dimensions: examples of decomposition of a linear representation in direct sum of sub-representations, Shurs lemma.

1.2 Finite dimensional vector spaces

- Finite dimensional vector spaces. Existence of basis, isomorphisms with Kn. Existence of supplement subspace. Rank of a linear function, rank of a system of vectors. Dual space, rank of a system of linear equations, transpose of a linear function. Dual basis. Biduality. Orthogonality.
- Multilinear functions. Determinant of a system of linear functions, of an endomorphism. Special linear group SL(E). Orientation of an R-vector space.
- Matrix with coefficients from a commutative ring. Elementary operations on rows and columns of a matrix, determinant, invertibility. Matrices with coefficients from a field. Matrix representations of a linear function. Change of basis,Gaussian elimination, row-echelon form. Solving systems of linear equations, calculating determinants, inverting square matrices, calculating the rank of a matrix, finding equations that define a vector subspace.
- Stable subspaces under endomorphism, kernel lemma. Characteristic polynomial, endomorphic polynomials. Annihilating polynomials. Minimal polynomials. Cayley-Hamilton theorem. Diagonalization, trigonalisation, characteristic subspaces, Dunford decomposition, exponentiation of real and complex matrices.

2 Groupes

The different notions of group theory introduced in the following paragraphs are to be applied in geometric situations.

- Groups, group morphisms. Direct group products. Subgroups, generative subgroups, order of an element, distinguished and normal subgroups, group quotients, group action on a set, stabilizers, orbites, quotient spaces, class formulas, conjugation class. Function to determine the isometric groups of a regular polytope in 2 and 3 dimensions

- Cyclic groups. Finite dimensional abelian groups. Groups with complex roots and nth of unity, primitive roots.
- Permutation groups (for finite sets). Decomposition of a permutation in product of transpositions, into products of cycles to support disjoints. Signature. Alternating groups. Determinants.
- Classic automorphic groups on a finite dimensional vector space. General linear group, special linear group, orthogonal group, special orthogonal group, unitary group, special unitary group.
- Representations of a finite group on a C-vector space. Abelian case. Orthogonality of irreducible characteristics. Dual group. Fourier transforms. Convolution. Maschke theorem. Characteristics of a finite dimensional representation. Central functions on the group, orthonormal basis of irreducible elements. Examples of low cardinality groups.

3 Rings, fields and polynomials

- Unitary rings, ring morphisms, subrings. The ring Zn on the integers. Ideal of a commutative ring, quotient rings, first ideals, maximal ideals. Chinese remainder theorem. Commutative ring algebra.
- Polynomial algebra on one or several indeterminates on a commutative ring. Roots of a polynomial, multiplicity. Relationship between the coefficients and the roots of a split polynomial. Newtonian sums. Derived polynomials. Decomposition into sums of homogeneous polynomials. Symmetric polynomials.
- Fields, subfields, Characteristics, Frobenius morphisms, field extensions, integral ring fractional fields. Field Q of rational numbers. Field R of real numbers. Field C of complex numbers. dAlembert-Gauss theorem. Algebraic elements transcendentals. Algebraic extensions. Closed algebraic fields. Rupture and decomposition fields. Finite fields.
- Divisibility in commutative integral rings. Irreducible elements, invertible elements, first elements among them. Factorial rings. Greatest common divisor, least common multiple. Factorability of A[X] when A is a factorial ring. Principle ring. Bezouts theorem. Euclidean rings. Euclids algorithm. Case of ring Z and K[X] algebra of polynomials on the field K. Irreducible polynomial. Cyclotomic polynomials in Q[X], Einsteins criteria.
- Congruence in Z. Prime numbers. Z/nZ and its invertible elements. Eurlers indicator function.
- Rational fraction fields with one indeterminate on the field. Decomposition into simple elements. Real and complex cases.

4 Bilieanar Forms and Quadratics on a Vector Space

- Bilinear forms. Alternating bilinear forms. Symmetric bilinear forms, quadratic forms, polar quadratic form (with characteristic difference of 2). Orthogonal elements, geometric interpretation. Non-degenerative forms. Adjoint to an endomorphism. Matrix representation. Change of basis. Rank of a bilinear form.
- Orthogonality. Subspace isotropes. Decomposition of a quadratic form into sums of squares. Sylvesters inertia theorem. Classification in the case of R or C. Orthogonalization process.
- Euclidian vector spaces, hermitian vector spaces. Isomorphisms of a Euclidian vector space with its dual. Orthogonal supplement. Cauchy-Schwarz inequality. Norm. Orthonormal basis.
- Orthogonal group, special orthogonal group. Examples of orthogonal group generators: decomposition by an orthogonal automorphism via reflexion products. Symmetric endomorphisms, normal endomorphisms. Diagonalisation of a symmetric endomorphism. Simultaneous reduction of two quadratic real forms, one being positive definite. Polar decomposition in GL(n,R). Euclidean vector spaces in 3 dimensions. Classification of elements in O(3,R). Mixed product, vector product.
- Unitary group, special unitary group. Diagonalization of normal endomorphisms. Polar decomposition in GL(n,C)

5 5 Gomtries affine et euclidienne

- All spaces considered in this section are finite dimensional
- Affine spaces and associated vector spaces. Affine functions and linear associated functions. Affine subspaces, barycenters. Affine spans , equations of affine subspaces. Affine groups, notions of affine properties. Homothetic transformation groups, affinity. Convex parts, convex envelopes of part of an affine real space. Extrema.
- Isometries of a Euclidian affine subspace. Euclidian affine space isometry group. Euclidean affine space isometries. Isometries that do and do not preserve orientation. Direct and indirect similarities of the plane. Isometric classification in two and three dimensions.
- Angles in two dimensions : angles of vectors, right angles, inscribed angle theorem, cocyclicity.
- Group of isometries that leave stable part of the plane or space. Regular polygons. Metric relations in the triangle. Using complex numbers in plane geometry.
- Application of quadratic forms to study proper conic sections of the affine euclidean plane (foyer, eccentricity) and of quadratics to study 3 dimensional euclidean affine spaces

6 Single Variable Real Analysis

6.1 Real Numbers

The field R of real numbers. Topology of R. Additive subgroups of R. Series of real numbers: convergence, adherence value, recurrent series. Limit infimum and supremum. Completeness of R. Bolzano-Weierstrass theorem. Compact parts of R. Caucy series. Connected parts of R.

6.2 Numerical Series

Convergence of real value series. Geometric series, Riemann series. Positive valued series. Summation of comparison relations. Comparison of a series and an integral. Error estimation. Absolute convergence. Products of series. Alternating series.

6.3 Real-valued functions defined on part of R

- Continuity Limites, continuity. Intermediate value theorem. Image of a segment. Study of continuity of monotonic functions. Continuity of reciprocal functions.
- Differentiability Derivative at a point, differentiable functions. Derivative of a composite function. Derivative of a reciprocal function. Rolles theorem and intermediate value theorem. Derivatives of higher order functions. Applications to Ck, to piecewise Ck. Leibniz formula. Taylors theorem in integral and differential form. Limit and asymptote calculations.

6.4 Usual functions (trig, rational, exp, log, etc).

Polynomial functions, rational functions. Logarithms. Exponential. Power functions. Circular and hyperbolic functions. Reciprocal circular and hyperbolic functions.

6.5 Integration

- Integral over a segment of piecewise continuous functions Calculation of primitives. Riemann sums. Primitives of a continuous function. Usual methods of calculating integrals. Change of variable. Integration by parts.
- Generalized integrals Absolutely convergent integrals. Integration of comparison relationships. Semi-converging integrals.

6.6 Sequences and series of functions

Simple convergence, uniform convergence. Continuity and differentiability of the limit. Case of series of functions; normal convergence. Polynomial Weierstrass and trigonometric Weierstrass approximation theorems.

6.7 Convexity

Convex functions of a real variable. Continuity and differentiability of the convex functions. Characterizations of convexity. Convexity inequalities.

7 Single Variable Complex Analysis

7.1 Complex Valued series

- Radius of convergence. Properties of sums of complex valued series on their disks of convergence. Continuity, differentiability with respect to the complex variable, primitives
- Complex exponentials. Extension of circular functions to the complex plane. Development of trig, exponential, log, and rational functions among others.

7.2 Functions on one complex variable

- Holomorphic functions. Cauchy-Riemann conditions. Line integrals of continuous functions in C. Primitives of a holomorphic function. Representations of the log function on C. Theorem of holomorphic functions under integral domains
- Winding number of a closed curve in C with respect to a point
- Cauchy formulas. Analyticity of a holomorphic function. Principle of isolated zeros. Cauchy formulas. Analyticity of a holomorphic function. Principle of analytic continuation. Maximum principle.
- Isolated singularities. Laurent series. Meromorphic functions. Residue theorem.
- Sequences and series of holomorphic functions. Holomorphic stability by uniform convergence.

8 Topology

8.1 Topology and Metric Spaces

- Topology of a metric space. Induced topology. Finite product of metric spaces.
- Series. Limit/cluster/accumulation points. Limits. Continuous functions. Homeomorphisms.
- Compactness. Equivalence of definitions in terms of cluster points (Bolzano-Weierstrass) or open covers (Borel-Lebesgue). Connectedness. Connected components. Arc connectedness.
- Lipschitz functions, uniformly continuous functions. Heine-Cantor theorem
- Complete metric spaces. Fixed point theorem for continuous functions.

8.2 Normed vector spaces on R and C

- Topology on a normed vector space. Equivalent norms. Finite dimensional spaces. Normes —— ——p on Rn and Cn. Banach spaces. Absolutely convergent series on Banach spaces
- Continuous linear functions, norme of a continuous linear function
- Norme of uniform convergence. Continuous bounded Banach space valued linear functions on a metric space.
- Compactness of parts of a normed vector space. Riesz Theorem, dAscolis Theorem.

8.3 Hilbert Spaces

- Hilbert projection theorem. Orthogonal projection onto closed vector subspaces.
- Dual space, Riesz representation theorem. l2 and L2 cases. Hilbert bases (in the separable case). Examples of trigonometric polynomial bases and orthogonal polynomials. Lax-Milgram theorem.
- H10(—0,1—) and its application to the Dirichlet problem in one dimension

9 Differential Calculus

9.1 Differential Calculus

- Differentiable functions on an open subset of Rn. Differentials (linear tangent functions). Derivatives with respect to a vector
- Partial derivatives. Jacobian matrix, gradient vector, Hessian matrix. Composition of differentiable functions. Mean value theorem. Differentiable functions
- Functions that can be differentiated k times. Kth partial derivative. Inversion of differentiation order. Taylors theorem in differential and integral form
- Local study of real valued functions. Series representations. Local extrema, concavity of functions on an open convex subset of Rn.
- Diffeomorphisms. Inverse function theorem. Implicit function theorem.

9.2 Differential equations

- differential equations of the form X = f(t, X) on I   where I is an open interval in R and Is an open subset of Rn. Cauchy-Lipschitz Theorem. Maximal solutions. Grnwall lemma. Exit theorem of a compact subspace
- Autonomous differential equations. Phase portraits, qualitative behavior. Stability of equilibrium points (linearisation theorem).
- Linear differential systems. Method of constant variation (Duhamels formula). Constant coefficient case. Solving systems of differential equations of order ¿ 1

9.3 9.3 Differential Geometry

- Generalizations to Rn. Equivalent definitions: local graphs, local parameterization, local equation, tangent space, gradient. R3 case, position with respect to the plane of the tangent.
- construction of curves/planes represented by a parametric equation. Metric study of curves, line integrals, curve length.
- Extrema, Lagrange multipliers.

10 Integral Calculus

10.1 Notions of measure theory

Definition of measurable spaces, sigma-algebras and their special cases, positive measure, special cases of counting measure, Lebesgue measure (construction allowed) and probability measures. Definition of a product measure (construction allowed). Definition of measurable functions, approximation by step functions.

10.2 Integration

- Integrals of positive measurable functions, monotonic conversion, Fanous lemma, integrable functions, dominated convergence theorem.
- finite dimensional vector-valued integrable functions, continuity, differentiability of parametric integrals
- L^p spaces where $1 \leq p \leq \infty$, completeness, Holders, inequality,
- Fubinis theorem, change of variables for multiple integrals, polar coordinate case, spherical coordinate case.
- Convolution. Regularization and approximation by regularization and approximation by convolution.

10.3 Fourier Analysis

- Fourier series of locally integrable periodic real-valued functions, Riemann-Lebesgue lemma, convolution product of periodic functions, Dirichlet, Fejer, and Parseval theorems
- Fourier transforms on $L^1(R^d)$ and $L^2(R^d)$, Plancherels theorem

11 Probabilities

11.1 Definitions of a probabilistic space

Events, measure of probability, independent events, sigma-algebra, 0-1 law, Borel-Cantelli lemma, conditional probability, total probability formulae

11.2 Random variables and their laws

- Discrete law, law of absolute continuity, probability density function, law of joint probability, independence of random variables, mean and variance of a real-valued random variable, transfer theorem, moments, examples of Bernoulli, binomial, geometrique, Poisson, uniform, exponential, and Gaussian distributions
- characteristic equation, generator functions, functions on sums of independent random variables.

11.3 Convergence of series of random variables

- Probabilistics convergence, in Lp, almost surely, markov inequality, Tchebychev inequality, Levys theorem
- Strong and weak laws of large numbers, central limit theorem.

12 Probability distribution calculus

For dimension $d > 1$, we assume the integration by parts formula $\int_\Omega \delta_x f(x)dx = \int_{d\Omega} f(x)\nu_j(x)d\sigma(x)$ On the domain $\Omega \subset R^d$, with border sufficiently regular, with $\delta\Omega$ being the measure of the Lebesgue on $\partial\Omega$ and v(x) being the unitary exterior vector on $x \in \partial\Omega$. On the other hand, it is expected to have a certain familiarity with manipulating such functions, for example in the case where Ω is a ball in R^d.

12.1 Distributions on Rd

- vector spaces on C with compact support, stability by derivation, stability by multiplication by a function on Coo, partitions of unity, constructing approximations of probability density functions in spaces of common functions (trig, exp, rational, log, etc).
- Distributions. Examples of distributions, locally integrable functions, dirac measures, Cauchy principal values, multiplication by a function in Coo, probability distribution function from a dataset, convergent distribution series, definitions, examples, notions of support for a distribution, punctual support

12.2 spaces $S(R^d)$ and $S(R^d)$

- Schwartz space S (Rd) of rapidly decreasing functions as well as all their derivatives, gaussian functions are their derivatives, stability by derivation, stability by multiplication by a function Coo of slow growth, Fourier transforms on S(Rd), convolution of two functions of S(Rd)
- Tempered distribution spaces in Spaces $S(R^d)$, linear forms of T on $S(R^d)$ such that there exists $C > 0$ and $k \in N$ such that $|hT|\phi i| \leq Csup|x\partial^{\beta\phi(x)}$—, xRd, $||k,||k$ forall S (Rd). examples of tempered distributions, functions L2 and Riesz representation, Lp functions, periodic case, Dirac comb, derivation of tempered distributions, multiplication by a function Cooof slow grows
- Fourier transforms on $S^0(Rd)$, inverse formula, Fourier transform and derivation, Fourier transform on a product of convolution.

12.3 Functions

Find the derivative and the Fourier transform of a distribution. Poissons formula, using convolution and Fourier-Laplace transform to solve one dimensional linear differential equations. Notion of constant coefficient differential operator elementary solution (laplacien case). Notion of weak solution of partial derivative equation. Functions, for example,solving the laplace equations, heat equations, wave equations.

13 Numerical methods

13.1 Solving systems of linear inequalities

Notion of conditioning. Gershgorin-Hadamard theorem. Gausss pivot, LU decomposition, iterative methods (for example Jacobi, gauss-seidel), convergence analysis, convergence analysis, spectral ray, singular value decomposition, example of discretisation matrix by finite differences of the laplacian in one dimension

13.2 iterative methods of solving systems of real and vector valued equations

linear systems case, iterative methods, proper element search, brute force method, optimization of convex function in finite dimension, gradient descent, square root, nonlinear problems with real and vector values, bisection method, Picard method, Newtons method, rate of convergence and estimation of error

13.3 Numerical integration

Rectangle method, error estimation. Monte-Carlo method: rate of convergence, application to the calculation of multiple integrals.

13.4 Approximation of numerical functions

Lagrange interpolation: Lagrange polynomial of a function at (n + 1) points, estimation of the error.

13.5 Ordinary differential equations

Numerical aspects of Cauchy's problem: explicit Euler method, consistency, stability, convergence, order.

13.6 Fourier transform

Discrete Fourier transform on a finite abelian group. Fast Fourier transform.