H. A. HELFGOTT

Appendix A. Bounding $\sum_{n\leq a} n^{-s} - \int_0^a \frac{du}{u^s}$, or approximating $\zeta(s)$

We want a good explicit estimate on

$$\sum_{n\leq a}\frac{1}{n^s} - \int_0^a \frac{du}{u^s},$$

for a a half-integer. As it turns out, this is the same problem as that of approximating $\zeta(s)$ by a sum $\sum_{n\leq a} n^{-s}$. This is one of the two[1] main, standard ways of approximating $\zeta(s)$.

The non-explicit version of the result was first proved in [HL21, Lemmas 1 and 2]. The proof there uses first-order Euler-Maclaurin combined with a decomposition of $\lfloor x\rfloor - x + 1/2$ that turns out to be equivalent to Poisson summation. The exposition in [Tit86, §4.7–4.11] uses first-order Euler-Maclaurin and van de Corput's Process B; the main idea of the latter is Poisson summation.

There are already several explicit versions of the result in the literature. In [Che99], [Kad13] and [Sim20], what we have is successively sharper explicit versions of Hardy and Littlewood's original proof. The proof in [DHZA22, Lemma 2.10] proceeds simply by a careful estimation of the terms in high-order Euler-Maclaurin; it does not use Poisson summation. Finally, [dR24] is an explicit version of [Tit86, §4.7–4.11]; it gives a weaker bound than [Sim20] or [DHZA22]. The strongest of these results is [Sim20].

We will give another version here, in part because we wish to relax conditions – we will work with $|\Im s| < 2\pi a$ rather than $|\Im s| \leq a$ – and in part to show that one can prove an asymptotically optimal result easily and concisely. We will use first-order Euler-Maclaurin and Poisson summation. We assume that a is a half-integer; if one inserts the same assumption into [DHZA22, Lemma 2.10], one can improve the result there, yielding an error term closer to the one here.

Notation. We recall that $e(\alpha) = e^{2\pi i\alpha}$, and $O^*(R)$ means a quantity of absolute value at most R.

A.1. The decay of a Fourier transform.

Our first objective will be to estimate the Fourier transform of $t^{-s}\mathbb{1}_{[a,b]}$. In particular, we will show that, if a and b are half-integers, the Fourier cosine transform has quadratic decay *when evaluated at integers*. In general, for real arguments, the Fourier transform of a discontinuous function such as $t^{-s}\mathbb{1}_{[a,b]}$ does not have quadratic decay.

Lemma A.1. *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R}\setminus\{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then*

$$\int_a^b t^{-s} e(\nu t)dt = \left.\frac{t^{-\sigma}e(\varphi_\nu(t))}{2\pi i\varphi_\nu'(t)}\right|_a^b + \sigma\int_a^b \frac{t^{-\sigma-1}}{2\pi i\varphi_\nu'(t)} e(\varphi_\nu(t))dt + \int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i(\varphi_\nu'(t))^2} e(\varphi_\nu(t))dt, \quad \text{(A.1)}$$

where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$.

Proof. We write $t^{-s}e(\nu t) = t^{-\sigma}e(\varphi_\nu(t))$ and integrate by parts with $u = t^{-\sigma}/(2\pi i\varphi_\nu'(t))$, $v = e(\varphi_\nu(t))$. Here $\varphi_\nu'(t) = \nu - \tau/(2\pi t) \neq 0$ for $t \in [a,b]$ because $t \geq a > |\tau|/(2\pi|\nu|)$ implies

Date: January 10, 2026.

[1]The other one is the approximate functional equation.

$|\nu| > |\tau|/(2\pi t)$. Clearly

$$udv = \frac{t^{-\sigma}}{2\pi i \varphi_\nu'(t)} \cdot 2\pi i \varphi_\nu'(t) e(\varphi_\nu(t)) dt = t^{-\sigma} e(\varphi_\nu(t))dt,$$

while

$$du = \left(\frac{-\sigma t^{-\sigma-1}}{2\pi i \varphi_\nu'(t)} - \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i (\varphi_\nu'(t))^2}\right) dt.$$

□

Lemma A.2. *Let $g : [a,b] \to \mathbb{R}$ be continuous, with $|g(t)|$ non-increasing. Then g is monotone, and $\|g\|_{\mathrm{TV}} = |g(a)| - |g(b)|$.*

Proof. Suppose g changed sign: $g(a') > 0 > g(b')$ or $g(a') < 0 < g(b')$ for some $a \leq a' < b' \leq b$. By IVT, there would be an $r \in [a', b']$ such that $g(r) = 0$. Since $|g|$ is non-increasing, $g(b') = 0$; contradiction. So, g does not change sign: either $g \leq 0$ or $g \geq 0$.

Thus, there is an $\varepsilon \in \{-1, 1\}$ such that $g(t) = \varepsilon |g(t)|$ for all $t \in [a,b]$. Hence, g is monotone. Then $\|g\|_{\mathrm{TV}} = |g(a) - g(b)|$. Since $|g(a)| \geq |g(b)|$ and $g(a)$, $g(b)$ are either both non-positive or non-negative, $|g(a) - g(b)| = |g(a)| - |g(b)|$. □

Lemma A.3. *Let $\varphi : [a,b] \to \mathbb{R}$ be C^1 with $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Let $h : [a,b] \to \mathbb{R}$ be such that $g(t) = h(t)/\varphi'(t)$ is continuous and $|g(t)|$ is non-increasing. Then*

$$\left|\int_a^b h(t) e(\varphi(t)) dt\right| \leq \frac{|g(a)|}{\pi}.$$

This is a statement of known type ("non-stationary phase").

Proof. Since φ is C^1, $e(\varphi(t))$ is C^1, and $h(t)e(\varphi(t)) = \frac{h(t)}{2\pi i \varphi'(t)} \frac{d}{dt} e(\varphi(t))$ everywhere. By Lemma A.2, g is of bounded variation. Hence, we can integrate by parts:

$$\int_a^b h(t) e(\varphi(t)) dt = \left.\frac{h(t) e(\varphi(t))}{2\pi i \varphi'(t)}\right|_a^b - \int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i \varphi'(t)}\right).$$

The first term on the right has absolute value $\leq \frac{|g(a)|+|g(b)|}{2\pi}$. Again by Lemma A.2,

$$\left|\int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i \varphi'(t)}\right)\right| \leq \frac{1}{2\pi} \|g\|_{\mathrm{TV}} = \frac{|g(a)| - |g(b)|}{2\pi}.$$

We are done by $\frac{|g(a)|+|g(b)|}{2\pi} + \frac{|g(a)|-|g(b)|}{2\pi} = \frac{|g(a)|}{\pi}$. □

Lemma A.4. *Let $\sigma \geq 0$, $\tau \in \mathbb{R}$, $\nu \in \mathbb{R} \setminus \{0\}$. Let $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then, for any $k \geq 1$, $f(t) = t^{-\sigma-k} |2\pi\nu - \tau/t|^{-k-1}$ is decreasing on $[a,b]$.*

Proof. Let $a \leq t \leq b$. Since $\left|\frac{\tau}{t\nu}\right| < 2\pi$, we see that $2\pi - \frac{\tau}{\nu t} > 0$, and so $|2\pi\nu - \tau/t|^{-k-1} = |\nu|^{-k-1} \left(2\pi - \frac{\tau}{t\nu}\right)^{-k-1}$. Now we take logarithmic derivatives:

$$t(\log f(t))' = -(\sigma + k) - (k+1) \frac{\tau/t}{2\pi\nu - \tau/t} = -\sigma - \frac{2\pi k + \frac{\tau}{t\nu}}{2\pi - \frac{\tau}{t\nu}} < -\sigma \leq 0,$$

since, again by $\frac{|\tau|}{t|\nu|} < 2\pi$ and $k \geq 1$, we have $2\pi k + \frac{\tau}{t\nu} > 0$, and, as we said, $2\pi - \frac{\tau}{t\nu} > 0$. □

Lemma A.5. *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R} \setminus \{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then*

$$\int_a^b t^{-s} e(\nu t) dt = \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b + \frac{a^{-\sigma-1}}{2\pi^2} O^*\left(\frac{\sigma}{(\nu - \vartheta)^2} + \frac{|\vartheta|}{|\nu - \vartheta|^3}\right),$$

where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi} \log t$ and $\vartheta = \frac{\tau}{2\pi a}$.

Proof. Apply Lemma A.1. Since $\varphi_\nu'(t) = \nu - \tau/(2\pi t)$, we know by Lemma A.4 (with $k = 1$) that $g_1(t) = \frac{t^{-\sigma-1}}{(\varphi_\nu'(t))^2}$ is decreasing on $[a, b]$. We know that $\varphi_\nu'(t) \neq 0$ for $t \geq a$ by $a > \frac{|\tau|}{2\pi|\nu|}$, and so we also know that $g_1(t)$ is continuous for $t \geq a$. Hence, by Lemma A.3,

$$\left|\int_a^b \frac{t^{-\sigma-1}}{2\pi i \varphi_\nu'(t)} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi} \cdot \frac{|g_1(a)|}{\pi} = \frac{1}{2\pi^2} \frac{a^{-\sigma-1}}{|\nu - \vartheta|^2},$$

since $\varphi_\nu'(a) = \nu - \vartheta$. We remember to include the factor of σ in front of an integral in (A.1).

Since $\varphi_\nu'(t)$ is as above and $\varphi_\nu''(t) = \tau/(2\pi t^2)$, we know by Lemma A.4 (with $k = 2$) that $g_2(t) = \frac{t^{-\sigma}|\varphi_\nu''(t)|}{|\varphi_\nu'(t)|^3} = \frac{|\tau|}{2\pi} \frac{t^{-\sigma-2}}{|\varphi_\nu'(t)|^3}$ is decreasing on $[a, b]$; we also know, as before, that $g_2(t)$ is continuous. Hence, again by Lemma A.3,

$$\left|\int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i (\varphi_\nu'(t))^2} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi} \frac{|g_2(a)|}{\pi} = \frac{1}{2\pi^2} \frac{a^{-\sigma-1}|\vartheta|}{|\nu - \vartheta|^3}.$$

□

Lemma A.6. *Let $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. Let $n \in \mathbb{Z}_{>0}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi n}$. Write $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi} \log t$. Then*

$$\frac{1}{2} \sum_{\nu = \pm n} \frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\Bigg|_a^b = \frac{(-1)^n t^{-s} \cdot \frac{\tau}{2\pi t}}{2\pi i \left(n^2 - \left(\frac{\tau}{2\pi t}\right)^2\right)}\Bigg|_a^b.$$

It is this easy step that gives us quadratic decay on n. It is just as in the proof of van der Corput's Process B in, say, [Ten15, I.6.3, Thm. 4].

Proof. Since $e(\varphi_\nu(t)) = e(\nu t) t^{-i\tau} = (-1)^\nu t^{-i\tau}$ for any half-integer t and any integer ν,

$$\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\Bigg|_a^b = \frac{(-1)^\nu t^{-s}}{2\pi i \varphi_\nu'(t)}\Bigg|_a^b$$

for $\nu = \pm n$. Clearly $(-1)^\nu = (-1)^n$. Since $\varphi_\nu'(t) = \nu - \alpha$ for $\alpha = \frac{\tau}{2\pi t}$,

$$\frac{1}{2} \sum_{\nu = \pm n} \frac{1}{\varphi_\nu'(t)} = \frac{1/2}{n - \alpha} + \frac{1/2}{-n - \alpha} = \frac{-\alpha}{\alpha^2 - n^2} = \frac{\alpha}{n^2 - \alpha^2}.$$

□

Proposition A.7. *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi}$. Write $\vartheta = \frac{\tau}{2\pi a}$. Then, for any integer $n \geq 1$,*

$$\int_a^b t^{-s} \cos 2\pi n t \, dt = \left(\frac{(-1)^n t^{-s}}{2\pi i} \cdot \frac{\frac{\tau}{2\pi t}}{n^2 - \left(\frac{\tau}{2\pi t}\right)^2}\right)\Bigg|_a^b + \frac{a^{-\sigma-1}}{4\pi^2} O^*\left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} + \frac{|\vartheta|}{|n-\vartheta|^3} + \frac{|\vartheta|}{|n+\vartheta|^3}\right).$$

Proof. Write $\cos 2\pi n t = \frac{1}{2}(e(nt) + e(-nt))$. Since $n \geq 1$ and $a > \frac{|\tau|}{2\pi}$, we know that $a > \frac{|\tau|}{2\pi n}$, and so we can apply Lemma A.5 with $\nu = \pm n$. We then apply Lemma A.6 to combine the boundary contributions $|_a^b$ for $\nu = \pm n$. □