# Spectra of equational laws

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based on discussions of the Equational Theories Project

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## Contents

Here we discuss four problems related to the spectrum of a law, namely to the set of cardinalities of (non-empty) models of the law. For instance, a law that is equivalent to law 2 ($x = y$) has spectrum $\{1\}$. Since the direct product of magmas satisfying a law also does so, the spectrum is stable under multiplication, and includes all infinite cardinalities as soon as it contains any value $> 1$. We thus focus on finite cardinalities and discuss the spectrum as being a multiplicative subset of $\mathbb{Z}_{>0}$. The four problems of interest are as follows.

- In section 1, finding which laws have full spectrum $\mathbb{Z}_{>0}$.
- In section 2, finding the spectrum when it is not $\mathbb{Z}_{>0}$.
- In section 3, finding the finite cardinalities of directly-irreducible, subdirectly-irreducible, and simple magmas satisfying a law.
- In section 4, counting magmas of a given cardinality satisfying a law. This leads to a notion of phases (vacuum, gas, liquid, solid, crystal).

We perform these investigations for laws up to different orders depending on the problem, as they present very different challenges. Within each section we start with a summary of results and of some open questions that seem within reach of short dedicated efforts.

# 1 Laws with a full spectrum

We describe here our investigation of which laws have full spectrum. As summarized in Table 1, we fully determine the answer for laws up to order 5, and fail for only 0.003% of the roughly six billion laws of order up to 9. Surprisingly, among laws that we could treat, the spectrum is full if and only if it contains $\{2, 3\}$ (hence $\{1, 2, 3, 4\}$). This leads to a first question.

**Open Question 1.1.** *Are there equations whose spectrum contains* $[\![1, 4]\!]$ *but not* $5$*? What are the lowest-order equations with that property? Same questions for equations whose spectrum contains* $[\![1, p-1]\!]$ *but not* $p$ *for each prime* $p$*.*

In subsection 1.1 we explain how the problem was quickly dispatched up to order 4, then in subsection 1.2 a more conceptual reason why having a model of size 2 often leads to a full spectrum. This allows us to focus on laws with certain shapes and rhyme schemes for which size-2 models are inconclusive. We explain in subsection 1.3 how 3-element magmas guide the construction of magmas of all odd sizes that are likely to satisfy the given law. For one law of order 5, subsection 1.4 provides an ad-hoc family of models of all sizes. Finally, subsection 1.5 explains how the final search was performed using a python wrapper around the Mace4 program.

Table 1: Number of laws of order 0–9 whose spectrum is known to be full (by linear or piecewise or ad-hoc models), or non-full (by the absence of model of size 2 or 3). Out of 5 996 643 396 laws of order up to 9, 3 607 005 194 (60.150%) have full spectrum, 2 389 459 735 (39.847%) do not, and 178 467 (0.003%) remain unknown. Laws of odd orders are comparatively better resolved because they cannot be obeyed by addition modulo 2, so more laws fail to have a 2-element model.

| Order | Full spectrum | | | | Non-full spectrum | | | Unknown |
|---|---|---|---|---|---|---|---|---|
| | total = | linear + | pieces + | ... | total = | no size 2 + | no size 3 | |
| 0 | 1 = | 1 + | 0 + | 0 | 1 = | 1 + | 0 | 0 |
| 1 | 3 = | 3 + | 0 + | 0 | 2 = | 2 + | 0 | 0 |
| 2 | 27 = | 27 + | 0 + | 0 | 12 = | 12 + | 0 | 0 |
| 3 | 229 = | 229 + | 0 + | 0 | 135 = | 135 + | 0 | 0 |
| 4 | 2814 = | 2808 + | 4 + | 2 | 1470 = | 1408 + | 62 | 0 |
| 5 | 35950 = | 35916 + | 32 + | 2 | 21932 = | 21920 + | 12 | 0 |
| 6 | 558125 = | 557137 + | 978 + | 10 | 329940 = | 315280 + | 14660 | 300 |
| 7 | 9227082 = | 9223929 + | 3153 + | 34 | 5892466 = | 5889143 + | 3323 | 523 |
| 8 | 172221522 = | 172064190 + | 157332 + | 0 | 109694882 = | 105709286 + | 3985596 | 25086 |
| 9 | 3424959441 = | 3424347706 + | 611735 + | 0 | 2273518895 = | 2272623385 + | 895510 | 152524 |

## 1.1 Laws up to order 4

Consider laws of order up to 4, modulo duality and equivalence, and denote these 741 classes by the lowest-numbered equation. We list them in Table 2 for reference. All laws obeyed by constant laws, or left/right projection, or by $x \diamond y = \pm x \pm y$ on $\mathbb{Z}/n\mathbb{Z}$ have full spectrum. This leaves 48 laws (and their duals and equivalents) that are not given a full spectrum by these considerations:

{2, 63, 66, 73, 115, 118, 125, 167, 168, 467, 474, 481, 501, 546, 556, 667, 670, 677, 695, 704, 873, 880, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1323, 1480, 1482, 1483, 1485, 1486, 1489, 1496, 1516, 1523, 1526, 1682, 1685, 1692, 1719}

For laws 1482, 1523, 1682 we have found models of all sizes: for law 1482, the model found by Douglas McNeil is $x \diamond y = 0$ except $0 \diamond 0 = 1$ and $0 \diamond x = x \diamond 0 = x$ for $x \neq 0, 1$; for law 1523, models with $x \diamond x = 0$, $x \diamond 0 = 0 \diamond x = x$; for law 1682, the model found by Zoltan Kocsis on any interval $[\![0, n-1]\!]$ is defined by

$$i \diamond j = \begin{cases} i & \text{if } j = 0, \\ \delta_{i=0} & \text{if } j = 1, \\ 1 + \delta_{j \text{ odd}} & \text{if } i = 0 \text{ and } j \geq 2, \\ j & \text{if } i = 1 \text{ and } j \geq 3 \text{ odd}, \\ i + \delta_{i<j} & \text{if } i, j \geq 2 \text{ and } i + j \text{ even}, \\ 2\delta_{i=1} + 0 & \text{if } i \geq 1 \text{ odd and } j = 2, \\ j - \delta_{i \geq j} & \text{if } i \geq 1 \text{ and } j \geq 3 \text{ and } i + j \text{ odd}. \end{cases} \tag{1}$$

For the other laws in the list, we show

Table 2: List of representatives of equivalence classes modulo duality.

{1, 2, 3, 4, 8, 9, 10, 11, 13, 14, 16, 38, 40, 41, 43, 47, 48, 49, 50, 52, 53, 55, 56, 58, 62, 63, 65, 66, 72, 73, 75, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 111, 115, 117, 118, 124, 125, 127, 138, 151, 152, 153, 159, 162, 167, 168, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 323, 325, 326, 327, 329, 332, 333, 335, 336, 343, 411, 412, 413, 414, 416, 417, 418, 419, 420, 422, 426, 427, 428, 429, 430, 432, 433, 434, 436, 437, 439, 440, 442, 443, 446, 450, 452, 455, 463, 464, 466, 467, 473, 474, 476, 477, 481, 492, 500, 501, 503, 504, 508, 510, 511, 513, 543, 546, 556, 562, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 626, 629, 630, 632, 633, 635, 639, 640, 642, 643, 645, 646, 647, 653, 655, 657, 658, 667, 669, 670, 676, 677, 679, 680, 690, 692, 695, 703, 704, 706, 707, 713, 714, 716, 723, 727, 731, 765, 778, 817, 818, 819, 820, 822, 823, 824, 825, 826, 827, 828, 829, 832, 833, 834, 835, 836, 837, 838, 839, 840, 842, 843, 844, 845, 846, 847, 848, 854, 856, 860, 861, 870, 872, 873, 879, 880, 882, 883, 887, 895, 898, 906, 907, 910, 916, 917, 947, 960, 978, 1020, 1021, 1022, 1023, 1025, 1026, 1027, 1028, 1029, 1032, 1033, 1035, 1036, 1037, 1038, 1039, 1041, 1042, 1043, 1045, 1046, 1048, 1049, 1050, 1051, 1052, 1053, 1055, 1056, 1060, 1061, 1063, 1073, 1075, 1076, 1082, 1083, 1085, 1086, 1096, 1109, 1110, 1112, 1113, 1117, 1119, 1122, 1133, 1137, 1167, 1171, 1184, 1223, 1224, 1225, 1226, 1227, 1228, 1229, 1230, 1231, 1232, 1233, 1234, 1235, 1236, 1237, 1238, 1239, 1240, 1241, 1242, 1243, 1244, 1245, 1248, 1249, 1250, 1251, 1252, 1253, 1254, 1255, 1256, 1259, 1262, 1263, 1264, 1267, 1271, 1276, 1278, 1279, 1285, 1286, 1288, 1289, 1312, 1313, 1315, 1316, 1322, 1323, 1325, 1340, 1353, 1370, 1374, 1387, 1426, 1427, 1428, 1429, 1431, 1432, 1434, 1435, 1437, 1441, 1442, 1443, 1444, 1445, 1446, 1447, 1448, 1451, 1453, 1454, 1457, 1461, 1465, 1469, 1478, 1479, 1480, 1481, 1482, 1483, 1484, 1485, 1486, 1488, 1489, 1491, 1492, 1496, 1515, 1516, 1518, 1519, 1523, 1525, 1526, 1586, 1629, 1630, 1631, 1632, 1633, 1634, 1635, 1636, 1637, 1638, 1641, 1644, 1645, 1647, 1648, 1650, 1654, 1655, 1657, 1660, 1664, 1672, 1681, 1682, 1684, 1685, 1687, 1691, 1692, 1694, 1695, 1701, 1718, 1719, 1721, 1722, 1724, 1728, 1729, 1731, 1738, 1793, 3253, 3254, 3255, 3256, 3257, 3258, 3259, 3260, 3261, 3262, 3263, 3264, 3265, 3267, 3268, 3269, 3270, 3271, 3272, 3273, 3274, 3275, 3277, 3278, 3279, 3280, 3281, 3282, 3283, 3284, 3285, 3288, 3290, 3292, 3294, 3296, 3297, 3300, 3306, 3308, 3309, 3312, 3315, 3316, 3317, 3318, 3319, 3320, 3321, 3322, 3323, 3326, 3331, 3334, 3342, 3343, 3345, 3346, 3349, 3350, 3352, 3353, 3355, 3363, 3364, 3385, 3388, 3398, 3414, 3417, 3456, 3457, 3458, 3459, 3460, 3461, 3462, 3463, 3464, 3465, 3466, 3467, 3468, 3469, 3470, 3471, 3472, 3473, 3474, 3475, 3476, 3477, 3478, 3479, 3480, 3481, 3482, 3483, 3484, 3485, 3487, 3488, 3489, 3491, 3493, 3495, 3496, 3497, 3499, 3503, 3509, 3511, 3512, 3513, 3515, 3518, 3519, 3520, 3521, 3522, 3523, 3524, 3525, 3526, 3527, 3529, 3532, 3533, 3534, 3537, 3541, 3545, 3546, 3548, 3549, 3555, 3556, 3558, 3566, 3583, 3587, 3588, 3591, 3600, 3601, 3607, 3617, 3620, 3634, 3659, 3660, 3661, 3662, 3663, 3665, 3666, 3667, 3668, 3669, 3670, 3671, 3672, 3673, 3675, 3676, 3678, 3679, 3681, 3682, 3683, 3703, 3712, 3714, 3715, 3716, 3718, 3721, 3722, 3723, 3724, 3726, 3727, 3728, 3729, 3730, 3735, 3737, 3740, 3744, 3748, 3751, 3756, 4268, 4269, 4270, 4271, 4272, 4273, 4274, 4275, 4276, 4277, 4278, 4279, 4280, 4283, 4284, 4286, 4287, 4288, 4290, 4291, 4293, 4295, 4296, 4297, 4299, 4300, 4301, 4303, 4304, 4305, 4314, 4315, 4318, 4320, 4321, 4325, 4327, 4331, 4343, 4358, 4362, 4364, 4369, 4380, 4381, 4382, 4383, 4384, 4385, 4386, 4387, 4388, 4389, 4390, 4391, 4392, 4393, 4396, 4397, 4398, 4399, 4400, 4401, 4402, 4403, 4404, 4405, 4406, 4407, 4408, 4410, 4411, 4412, 4413, 4415, 4416, 4417, 4421, 4423, 4424, 4428, 4430, 4433, 4434, 4435, 4437, 4438, 4439, 4441, 4443, 4444, 4445, 4447, 4448, 4449, 4456, 4458, 4460, 4461, 4470, 4471, 4474, 4476, 4478, 4481, 4482, 4484, 4485, 4490, 4497, 4502, 4512, 4513, 4515, 4517, 4519, 4520, 4526, 4531, 4535, 4541, 4544}

- The 31 laws {2, 63, 66, 73, 115, 118, 125, 167, 168, 467, 474, 501, 670, 677, 704, 873, 880, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1480, 1486, 1489, 1516, 1685, 1692, 1719} have no model of size 2.
- The 14 laws {481, 546, 556, 667, 695, 883, 887, 895, 898, 1323, 1483, 1485, 1496, 1526} have models of size 2 but none of size 3.

These $3+31+14 = 48$ laws cover the whole list above. Curiously, this situation persists to higher orders: laws that have models of size 2 and 3 appear to have full spectrum in all cases we were able to resolve.

## 1.2 Full spectrum from 2-element magmas

Let us consider a law $A \geq 2$ (to exclude the trivial cases of law 1 and law 2) that has a model on $\{0, 1\}$. There are 10 magmas of that size up to isomorphism, which we can denote by their multiplication table, such as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ corresponding to addition modulo 2. For many of these models, laws satisfying them must have quite a constrained shape or rhyme scheme, which ensures they have full spectrum.

- If $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ obeys the law then the left and right-hand sides must have at least one occurrence of the operation (otherwise the equation would fail when all variables are 1), and the law is satisfied by constant operations $x \diamond y = 0$ on sets of any size. We henceforth assume that the law is *not* obeyed by this 2-element model, hence that it has the form $x = \dots$.

- If $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)$ obeys the law, then the first variable of the right-hand side has to be $x$: indeed, if it is some other variable $y$ then taking $y = 0$ we see that the right-hand side equals 0 independently of $x$ because of $0 \diamond \text{anything} = 0$. Laws whose sides both start with the same variable have full spectrum thanks to the left projection operation $x \diamond y = x$ on a set of any size. Dually, if $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$ obeys the law then the last variable in the law is $x$ and right-projection magmas obey the law, giving it a full spectrum. We henceforth assume that the first and last variables of the law differ from $x$.

- If $\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right)$ obeys the law then $x \diamond y$ only depends on $y$, so the law's right-hand side only depends on the last variable, which must be $x$, so actually the right-projection magma obeys the law and we are in the previous situation. Likewise if $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)$ obeys the law.

Finally, if the only models of size 2 are one of the remaining two models $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and/or $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, then the spectrum is not guaranteed to be full. The first of these models is addition modulo 2, obeyed precisely by the laws whose number of $x$, number of $y$ etc on the two sides coincide modulo 2. This happens only for laws of even order. The second model is the Nand logic gate. Laws obeyed by this magma have quite constrained shapes: indeed, setting all variables to be equal we get a one-variable law that must be satisfied. The lowest-order shapes obeyed by Nand are

$$\begin{aligned} x, \quad (x \diamond x) \diamond (x \diamond x), \quad &((x \diamond x) \diamond x) \diamond (x \diamond x), \quad (x \diamond x) \diamond ((x \diamond x) \diamond x), \\ &(x \diamond (x \diamond x)) \diamond (x \diamond x), \quad (x \diamond x) \diamond (x \diamond (x \diamond x)). \end{aligned} \tag{2}$$

A more conceptual view on why some 2-element magmas lead to a full spectrum is that powers of these 2-element magmas contain submagmas of all sizes. These submagmas are also called subdirect products. In contrast, powers of the "addition modulo 2" magma are Boolean groups, whose submagmas are also Boolean groups, of cardinality $2^k$, $k \geq 0$, and likewise submagmas of powers of Nand are Boolean algebras (equipped with the Sheffer stroke) and have cardinality $2^k$, $k \geq 0$.

We could try to pursue the logic of the previous section for 3-element magmas. There are 3330 isomorphism classes of 3-element magmas, studied by Berman and Burris [?] who developped a numbering scheme, the "Siena Catalog" (see a discussion on Zulip). This may help resolve 1.1. However, we have been quite successful with a slightly different way of using 3-element magmas.

## 1.3 Piecewise models from 3-element magmas

The techniques described so far (linear models, versus absence of models of sizes 2 or 3) resolve all laws of order 3, all laws of order 4 except for {1482, 1523, 1682, 1885, 2125, 2132}, and all laws of order 5 except for 34 laws. Since these laws have models of size 2, it it enough (by multiplicativity of the spectrum) to find models of all odd sizes. Investigating some of the order-5 laws using Mace4 led to a model on $[\![-k, k]\!]$ with $x \diamond y = -y$ if $y > 0$ and $x \diamond y = -x$ if $x, y \leq 0$ and $x \diamond y = 0$ if $y \leq 0 \leq x$. This model satisfies 28 of the remaining order 5 laws, which thus have full spectrum. More importantly, the piecewise linear construction generalizes to models on $[\![-k, k]\!]$ in which $x \diamond y \in \{0, \pm x, \pm y\}$, with the choice depending only on the sign of $x, y$ and ordering of $|x|, |y|$. Candidate models for a given law are built as follows.

- Consider models of the law on $\{-1, 0, 1\}$ with $0 \diamond 0 = 0$.
- For each such model, seek a model on $\{-2, -1, 0, 1, 2\}$ such that $\{-1, 0, 1\}$ and $\{-2, 0, 2\}$ are submagmas given by the chosen model, namely for $|x|, |y| \leq 1$ we take $x \diamond y$ to be as in the 3-element model, and $(2x) \diamond (2y) = 2(x \diamond y)$. There remains to choose the sixteen products $\pm 1 \diamond \pm 2$, $\pm 2 \diamond \pm 1$ in such a way that the law is satisfied.
- Extend uniquely the model to a model on $[\![-k, k]\!]$ by setting $x \diamond y \in \{0, \pm x, \pm y\}$ chosen such that each 5-element subset $\{0, \pm i, \pm j\}$ with $i < j$ has the same operation table. Then check that the law is obeyed. This can be done by checking concretely that the law is obeyed for $k$ equal to the total number of variables in the law.

This construction sometimes fails for some models of size 3 of a law and succeeds for others.

As summarized in Table 1, this technique is quite successful, as it dispatches laws {1482, 1523, 2125, 2132} and 32 of the laws of order 5, leaving us with only four laws up to order 5: law 1682 $x = (y \diamond x) \diamond ((x \diamond x) \diamond y)$, law 17191 $x = (y \diamond x) \diamond (x \diamond (y \diamond (y \diamond z)))$ and their duals. The first one has models of all sizes given by (1). The second one is now shown to have full spectrum, which fully resolves the question of interest up to order 5.

The techniques apply beyond order 5 of course, but the left-over laws become too numerous to consider one at a time. The linear and piecewise models resolve the question for all laws of order up to 6 except 314 of them (157 dual pairs), listed in Table 3, of which 7 dual pairs are shown to have full spectrum by the ad-hoc models of law 1682 and law 17191. This leaves 300 unknown laws. The first few are

$$\begin{aligned}
&\text{Law 71936: } x = y \diamond (x \diamond (y \diamond ((y \diamond y) \diamond (y \diamond y)))),\\
&\text{Law 72573: } x = y \diamond (y \diamond (y \diamond ((x \diamond y) \diamond (y \diamond y)))),\\
&\text{Law 76713: } x = y \diamond (y \diamond (y \diamond ((x \diamond (y \diamond y)) \diamond y))),\\
&\text{Law 80887: } x = y \diamond (y \diamond (y \diamond (((y \diamond y) \diamond x) \diamond y))),\\
&\text{Law 85027: } x = y \diamond (y \diamond ((y \diamond y) \diamond (y \diamond (x \diamond y)))).
\end{aligned} \tag{3}$$

Likewise, 17 dual pairs of laws of order 7 are consequences of 17191 hence are resolved by the ad-hoc model.

Table 3: List of laws (modulo duality) of order up to 6 whose spectrum contains 2 and 3 but is not known to be full by general techniques. Only one of each dual pair is listed. The underlined laws are those whose spectrum is known to be full thanks to ad-hoc models.

{1682, 17191, 71936, 72573, 76713, 80887, 85027, 89160, 93273, 97299, 97413, 109196, 109719, 109833, 109860, 109867, 113859, 118140, 121616, 125555, 125716, 125756, 126420, 126427, 129896, 138136, 138176, 138813, 138840, 138847, 142316, 163016, 163539, 163653, 163687, 171296, 171933, 175436, 175959, 176073, 176107, 179576, 180213, 184387, 191996, 200947, 205080, 209193, 213219, 217359, 217507, 221640, 224915, 225116, 225639, 225753, 225780, 225787, 229927, 234033, 234067, 237373, 237386, 237388, 237389, 237390, 237392, 237396, 237496, 238207, 241513, 241516, 241523, 241526, 241532, 241536, 241575, 241663, 242340, 245769, 245776, 245816, 246453, 246487, 249829, 249833, 249846, 249850, 249868, 250627, 253901, 253911, 253938, 253948, 254733, 254760, 258048, 258060, 262188, 262363, 267187, 274595, 274635, 275460, 278735, 279573, 282875, 282922, 283076, 283747, 287015, 287203, 287351, 287880, 291155, 291168, 291993, 292027, 295295, 295308, 295342, 295496, 299448, 299460, 300260, 300261, 307715, 307728, 307762, 307869, 307876, 307916, 311855, 312539, 312727, 316196, 316716, 320172, 320336, 320849, 320960, 320961, 324300, 324429, 324436, 324463, 324476, 325147, 325628, 328452, 328569, 329129, 329935, 332558, 332716, 332756, 333941}

## 1.4 An ad-hoc model for law 17191

Consider law 17191 $x = (y \diamond x) \diamond (x \diamond (y \diamond (y \diamond z)))$. Models of sizes 2 and 5 are easy to find using Mace4. Any other size belongs to some interval $[\![k(k+1)/2, k^2]\!]$, for some $k \geq 1$, and here is a family of models that have all theses sizes. This shows that law 17191 has full spectrum.

Let $E \subset [\![0, k-1]\!]^2$ be a collection of edges such that there does not exist $x, y$ with $(x, y), (y, x) \in E$, namely there are no self-loops $(x, x)$ nor cycle of length 2. Then define a magma operation on $M = [\![0, k-1]\!]^2 \setminus E$ by $(x, y) \diamond (z, w) = (y, z)$ whenever this is possible, and otherwise $(y, y)$ or $(z, z)$ as needed to make certain properties hold. Specifically, let

$$(x, y) \diamond (z, w) = \begin{cases} (y, y) & \text{if } (y, z) \in E, z = w, \\ (z, z) & \text{if } (y, z) \in E, z \neq w, \\ (y, z) & \text{if } (y, z) \notin E. \end{cases} \tag{4}$$

In particular $(x, y) \diamond (y, z) = (y, y)$. In the case $E = \emptyset$ this is just a normal central groupoid. More generally, we can evaluate

$$(x, y) \diamond \big((x, y) \diamond (z, w)\big) = \begin{cases} (x, y) \diamond (y, y) = (y, y) & \text{if } (y, z) \in E, z = w, \\ (x, y) \diamond (z, z) = (y, y) & \text{if } (y, z) \in E, z \neq w, \\ (x, y) \diamond (y, z) = (y, y) & \text{if } (y, z) \notin E. \end{cases} \tag{5}$$

The middle case is interesting because the condition $(y, z) \in E$ that caused $z$ to stick around after the first operation is responsible for eliminating it now. Altogether we have $(x, y) \diamond ((x, y) \diamond (z, w)) = (y, y)$, which simplifies equation 17191 to $(x, y) = ((z, w) \diamond (x, y)) \diamond ((x, y) \diamond (w, w))$. We compute, for $(x, y), (z, w) \in M$,

$$((z, w) \diamond (x, y)) \diamond ((x, y) \diamond (w, w)) = \Big((w, x) \text{ or } (x, x) \text{ or } (w, w)\Big) \diamond \Big((y, w) \text{ or } (y, y)\Big). \tag{6}$$

When the first operand is one of the first two cases $(\ldots, x)$, this correctly results in $(x, y)$ since $(x, y) \notin E$. The only danger is the last case $(w, w)$, which occurs only if $(w, x) \in E$ and $x = y$. In that case, $(x, w) \notin E$ (by the assumption on cycles) so the second operand is $(x, w)$. Then we compute $(w, w) \diamond (x, w) = (x, x)$ because $(w, x) \in E$ and $x \neq w$.

### 1.5 Some implementation details

We have implemented the techniques in a Python wrapper around many calls to Mace4. As we have explained in subsection 1.2, many laws can be immediately resolved according to their shape or rhyming scheme. These laws are not ever considered, but rather are directly counted in terms of Catalan and Bell numbers, among others. There are two remaining classes of laws to consider.

Firstly, laws obeyed by addition modulo 2 but not by any other 2-element magma except possibly Nand. These have any shape $x = \dots$ with trivial left-hand side, but their rhyme scheme is constrained to contain an even number of each variable. For each such law, we check whether it is obeyed by any of the operations $x \diamond y = \pm x \pm y$ for some choice of signs (then it has full spectrum), whether it fails to have a model of size 3 (using Mace4), and whether it has a piecewise linear model.

Secondly, laws obeyed by Nand but no other 2-element magma. These have very constrained shapes, and for each shape we go through possible rhyme schemes. For a pair of shape and rhyme scheme, we check whether Nand satisfies the law, then whether the law has any model of size 3, and whether a piecewise linear construction gives a model for all odd sizes, hence a full spectrum.

The order of checks affects strongly the computing costs. For orders up to 9, the code runs in a few hours on a consumer laptop.

## 2 Spectrum of laws of order up to 4

Build magmas obeying law 115 $x = y \diamond ((x \diamond x) \diamond y)$ (and as a result law 873) of all cardinalities $n \equiv 2 \mod 3$ other than $n = 2$, see subsection 2.2. Build magmas obeying law 1489 based on a graph theory description, see subsubsection 2.5.1. Find the spectrum of the Dupont law 63 $x = y \diamond (x \diamond (x \diamond y))$, and use the resulting techniques to resolve other equations in subsection 2.3.

We explain here in a streamlined way the results of the investigation discussed at `https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Order.203.20Spectra/near/526300502`. From the full-spectrum discussion, we already know that many laws have full spectrum, either through linear models, or through ad-hoc models for laws 1482, 1523,

1682. This leaves 45 laws up to duality and equivalence:

| | | | | | |
|---|---|---|---|---|---|
| 2 | $x = y$ | 667 | $x = y \diamond (x \diamond ((x \diamond x) \diamond y))$ | 1279 | $x = y \diamond (((x \diamond x) \diamond y) \diamond y)$ |
| 63 | $x = y \diamond (x \diamond (x \diamond y))$ | 670 | $x = y \diamond (x \diamond ((x \diamond y) \diamond y))$ | 1286 | $x = y \diamond (((x \diamond y) \diamond x) \diamond y)$ |
| 66 | $x = y \diamond (x \diamond (y \diamond y))$ | 677 | $x = y \diamond (x \diamond ((y \diamond x) \diamond y))$ | 1313 | $x = y \diamond (((y \diamond x) \diamond x) \diamond y)$ |
| 73 | $x = y \diamond (y \diamond (x \diamond y))$ | 695 | $x = y \diamond (x \diamond ((z \diamond z) \diamond y))$ | 1323 | $x = y \diamond (((y \diamond y) \diamond x) \diamond y)$ |
| 115 | $x = y \diamond ((x \diamond x) \diamond y)$ | 704 | $x = y \diamond (y \diamond ((x \diamond x) \diamond y))$ | 1480 | $x = (y \diamond x) \diamond (x \diamond (x \diamond z))$ |
| 118 | $x = y \diamond ((x \diamond y) \diamond y)$ | 873 | $x = y \diamond ((x \diamond x) \diamond (y \diamond y))$ | 1483 | $x = (y \diamond x) \diamond (x \diamond (y \diamond z))$ |
| 125 | $x = y \diamond ((y \diamond x) \diamond y)$ | 880 | $x = y \diamond ((x \diamond y) \diamond (x \diamond y))$ | 1485 | $x = (y \diamond x) \diamond (x \diamond (z \diamond y))$ |
| 167 | $x = (y \diamond x) \diamond (x \diamond y)$ | 883 | $x = y \diamond ((x \diamond y) \diamond (y \diamond y))$ | 1486 | $x = (y \diamond x) \diamond (x \diamond (z \diamond z))$ |
| 168 | $x = (y \diamond x) \diamond (x \diamond z)$ | 887 | $x = y \diamond ((x \diamond y) \diamond (z \diamond z))$ | 1489 | $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ |
| 467 | $x = y \diamond (x \diamond (x \diamond (y \diamond y)))$ | 895 | $x = y \diamond ((x \diamond z) \diamond (y \diamond z))$ | 1496 | $x = (y \diamond x) \diamond (y \diamond (z \diamond z))$ |
| 474 | $x = y \diamond (x \diamond (y \diamond (x \diamond y)))$ | 898 | $x = y \diamond ((x \diamond z) \diamond (z \diamond y))$ | 1516 | $x = (y \diamond y) \diamond (x \diamond (x \diamond y))$ |
| 481 | $x = y \diamond (x \diamond (y \diamond (z \diamond z)))$ | 907 | $x = y \diamond ((y \diamond x) \diamond (x \diamond y))$ | 1526 | $x = (y \diamond y) \diamond (y \diamond (x \diamond y))$ |
| 501 | $x = y \diamond (y \diamond (x \diamond (x \diamond y)))$ | 1076 | $x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$ | 1685 | $x = (y \diamond x) \diamond ((x \diamond y) \diamond y)$ |
| 546 | $x = y \diamond (z \diamond (x \diamond (z \diamond y)))$ | 1083 | $x = y \diamond ((x \diamond (y \diamond x)) \diamond y)$ | 1692 | $x = (y \diamond x) \diamond ((y \diamond x) \diamond y)$ |
| 556 | $x = y \diamond (z \diamond (y \diamond (x \diamond z)))$ | 1110 | $x = y \diamond ((y \diamond (x \diamond x)) \diamond y)$ | 1719 | $x = (y \diamond y) \diamond ((x \diamond x) \diamond y)$ |

Some of these laws are equivalent for finite magmas, specifically laws {63, 1692}, laws {73, 118*, 125}, laws {115, 880*}, laws {481, 1496*}, laws {883*, 1323, 1526} where * denotes the dual law. This reduces the number of laws to 38.

Many of these laws (for instance those of the form $x = y \diamond \dots$) imply that left multiplications are surjective. In a finite setting the left multiplications are thus bijective, namely the magma is a left quasigroup. There is then a left division operation uniquely defined by $x \diamond (x : y) = y$. For several pairs of laws, "$\diamond$" obeys one of the laws if and only if ":" obeys the other (or its dual), in other words the laws are parastrophically equivalent in finite magmas.[1] This means that the two equations share the same finite spectrum.[2] In this way we find

$$\mathrm{spec}(\text{E63}) = \mathrm{spec}(\text{E73}), \qquad \mathrm{spec}(\text{E546}) = \mathrm{spec}(\text{E556}). \tag{7}$$

Note that in a (left-)quasigroup obeying law 467 $x = y \diamond (x \diamond (x \diamond (y \diamond y)))$, the left division operation defined by $x \diamond (x : y) = y$ obeys law 437 $x = x : (y : (y : (x : y)))$, but we cannot make use of that because the models of law 437 that we used to show the full spectrum property are left-projections, which are maximally far from being left quasigroups. The same problem occurs for law 481. Laws 474 and 501 and 898 are parastrophically equivalent to themselves (or their dual). The other laws do not have a shape conducive to having a parastrophically equivalent law. We are left with 36 laws which could have distinct spectra, distinct from $\mathbb{Z}_{>0}$:

{2, 63, 66, 115, 167, 168, 467, 474, 481, 501, 546, 667, 670, 677, 695, 704, 873, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1480, 1483, 1485, 1486, 1489, 1516, 1685, 1719}

We organize the discussion around five classes of laws: related to abelian groups, to semi-symmetric quasigroups, to the Dupont law 63, to the central groupoid law 168, and some isolated laws. Before we start, we note the obvious fact

$$\mathrm{spec}(\text{E2}) = \{1\}. \tag{8}$$

[1] Normally parastrophic equivalence is defined for quasigroups and allows for both left and right division, and for the operations with operands swapped. In the context of left quasigroups, right division is not defined so only four of the six parastrophes are allowed.

[2] When discussing the (sub)directly irreducible and simple spectra there could be differences for infinite cardinalities, because the laws typically do not imply that infinite magmas are left quasigroups.

## 2.1 Laws from which a group structure is definable

First some laws related to abelian groups. Note that there are some more laws implying linearity, which have already been eliminated at an earlier stage by models of the form $x \diamond y = \pm x \pm y$, most crucially Tarski's law 543 defining abelian group subtraction.

546 $x = y \diamond (z \diamond (x \diamond (z \diamond y)))$ (and 556) has spectrum $\{k^2 + l^2 \mid k, l \in \mathbb{Z}\}$, namely all integers $n \geq 1$ whose valuation $v_p(n)$ is even for each prime $p \equiv 3 \mod 4$. For any fixed element $u$, the operation $x \ominus_u y := (x \diamond u) \diamond (u \diamond y)$ obeys the Tarski law 543, namely is group subtraction. With some more work one can show that $x \diamond y = -x + \iota(y) + c$ in some module over $\mathbb{Z}[\sqrt{-1}]$ with $\iota$ being multiplication by $\sqrt{-1}$. Thus the spectrum is that of abelian groups such that there exists an automorphism $\iota$ that squares to negation. Finite abelian groups are products of $p$-groups

$$M = \prod_{p \text{ prime}} M_p, \qquad M_p = \prod_{j=1}^{l_p} \mathbb{Z}/p^{m_j}\mathbb{Z}, \tag{9}$$

and the automorphism $\iota$ acts independently as automorphisms $\iota_p \colon M_p \to M_p$. Since $(\iota_p)^{\circ 4} = \mathrm{id}$, the orbits of $\iota_p$ have 4, 2, or 1 element. For $p \neq 2$, the square $\iota_p \circ \iota_p$ has a unique fixed point $0 \in M_p$ so all other orbits of $\iota_p$ have 4 elements and thus $|M_p| = p^{v_p(n)} \equiv 1 \mod 4$. This shows that $v_p(n)$ is even for any prime $p \equiv 3 \mod 4$. Conversely, constructing magmas for each value in the spectrum is immediate: $\mathbb{Z}[\sqrt{-1}]/(k + \sqrt{-1}l)\mathbb{Z}[\sqrt{-1}]$ has $k^2 + l^2$ elements.

895 $x = y \diamond ((x \diamond z) \diamond (y \diamond z))$ has spectrum $\{2^n \mid n \geq 0\}$. It characterizes Boolean groups (abelian groups of exponent 2).

898 $x = y \diamond ((x \diamond z) \diamond (z \diamond y))$ has spectrum $\{2^n \mid n \geq 0\}$. It is at least that because law 895 implies 898. To get the converse, we work out that the law 898 implies, for any fixed element $u$, that the operation $x \oplus_u y = ((u \diamond x) \diamond (y \diamond u)) \diamond u$ obeys law 895.

## 2.2 Laws related to semi-symmetric quasigroups 14

Next, let us discuss laws related to the semi-symmetric quasigroup law 14. Here, we call Mendelsohn quasigroup an idempotent semi-symmetric quasigroup (obeying law 4961, $x = y \diamond (x \diamond (y \diamond (z \diamond (y \diamond z))))$, or equivalently laws 3 and 14). They are in one-to-one correspondence with Mendelsohn triple systems on the same set so their spectrum is $\{0, 1 \mod 3\} \setminus \{6\}$.

Semi-symmetric loops (magmas obeying law 14 and 40) are in one-to-one correspondence with Mendelsohn triple systems on $M \setminus \{e\}$ so their spectrum is $\{1, 2 \mod 3\} \setminus \{7\}$.

66 $x = y \diamond (x \diamond (y \diamond y))$ has spectrum $\{0, 1 \mod 3\} \setminus \{6\}$. Such magmas are in one-to-one correspondence with Mendelsohn quasigroups equipped with

an involutive automorphism. In one direction, denote $S(x) = x \diamond x$ and verify that the operation $x \Box y = S(x \diamond y)$ obeys laws 3 and 14, and that $S(S(x)) = x$ and $S(x \diamond y) = S(x) \diamond S(y)$. In the other direction, verify that $x \diamond y = S(x \Box y)$ satisfies law 66 provided "$\Box$" obeys laws 3 and 14 and $S$ is an involutive automorphism. The spectrum is thus contained in that of Mendelsohn quasigroups, and since $S$ can be taken to be the identity the spectra are equal.

115 $x = y \diamond ((x \diamond x) \diamond y)$ (and 880) has **conjectural** spectrum $[\![1, +\infty) \setminus \{2, 6\}$. The law is obeyed in Mendelsohn quasigroups so its spectrum contains all $n \equiv 0, 1 \mod 3$ except 6. There remains to prove existence of models of all sizes $n = 3k + 2 \geq 5$ (showing it for $n = 8$, $n$ prime, and $n/2$ prime would be enough). An ATP run shows there are models at least up to order 53, in which the squaring map $S \colon x \mapsto x \diamond x$ has a large cycle of size $3k + 1$ (or possibly $3j + 1$ for $j$ close to $k$), making most of the magma translationally-invariant. For magma sizes up to 14, the possible cycle sizes for the squaring map (apart from having it be the identity) are $5 = 4{+}1$, $7 = 7$, $8 = 7{+}1$, $9 = 6{+}1{+}1{+}1$, $10 = 7{+}1{+}1{+}1$, $11 = 5{+}5{+}1$, $11 = 7{+}1{+}1{+}1{+}1$, $12 = 8{+}1{+}1{+}1{+}1$, $12 = 9{+}1{+}1{+}1$, $13 = 4{+}4{+}4{+}1$, $13 = 8 + 4 + 1$, $13 = 9 + 1 + 1 + 1 + 1$, $13 = 13$, $14 = 5 + 5 + 1 + 1 + 1 + 1$, $14 = 7 + 7$, $14 = 10 + 1 + 1 + 1 + 1$, $14 = 12 + 1 + 1$. There is probably some counting argument for which sizes are possible.

481 $x = y \diamond (x \diamond (y \diamond (z \diamond z)))$ (and 1496) has **conjectural** spectrum $[\![1, +\infty) \setminus \{3, 6\}$. It is equivalent to unipotence $x \diamond x = e$ and $x = y \diamond (x \diamond (y \diamond e))$. Finite magmas are quasigroups and the maps $x \mapsto x \diamond e$ and $x \mapsto e \diamond x$ are automorphisms, inverses of each other. The special case where these automorphisms are the identity is semi-symmetric loops. The spectrum thus contains $\{1, 2 \mod 3\} \setminus \{7\}$. It also contains $\{7, 9, 12\}$ by an ATP search. We see in these examples that the squaring map has specific cycle sizes. A natural conjecture is that the law is flexible enough to allow arbitrary magma sizes, except for the low-lying values 3,6.

501 $x = y \diamond (y \diamond (x \diamond (x \diamond y)))$ has **unknown** spectrum containing 1,4,5,8,9. It implies that the operation $x \Box y = x \diamond (x \diamond y)$ obeys law 14.

667 $x = y \diamond (x \diamond ((x \diamond x) \diamond y))$ has **unknown** spectrum containing $\{1, 2 \mod 3\} \cup \{9\} = \{1, 2, 4, 5, 7, 8, 9, 10, 11, \dots\}$. It is obeyed by semi-symmetric loops, hence the spectrum contains $\{1, 2 \mod 3\} \setminus \{7\}$, and by idempotent magmas satisfying the Dupont law 63 (see below), from which one gets 7 in the spectrum. An ATP run gives that 9 is also there.

695 $x = y \diamond (x \diamond ((z \diamond z) \diamond y))$ has spectrum $\{1, 2 \mod 3\} \setminus \{7\}$. Indeed, any semi-symmetric loop obeys law 695, and conversely the operation $x \Box y = (x \diamond x) \diamond (x \diamond y)$ obeys the semi-symmetric loop law 887.

873 $x = y \diamond ((x \diamond x) \diamond (y \diamond y))$ **conjectural** spectrum $[\![1, +\infty) \setminus \{2, 6\}$. The spectrum contains that of law 115 because the dual of law 115 implies law 873. The spectrum does not contain 2 and 6 thanks to an ATP run.

883 $x = y\diamond((x\diamond y)\diamond(y\diamond y))$ (and 1323, 1526) has **unknown** spectrum containing $\{1, 2 \mod 3\}$. It is obeyed by semi-symmetric loops *and* idempotent magmas obeying law 73 (parastrophically equivalent to the Dupont law). The size 7 model is such an idempotent magma.

887 $x = y\diamond((x\diamond y)\diamond(z\diamond z))$ has spectrum $\{1, 2 \mod 3\}\setminus\{7\}$. Indeed, the law characterizes exactly the semi-symmetric loops, equivalent to Mendelsohn triple systems on $M \setminus \{e\}$.

1083 $x = y \diamond ((x \diamond (y \diamond x)) \diamond y)$ has **unknown** spectrum containing 1,3,4,7,8,9. It implies that $x \square y = x \diamond (y \diamond x)$ obeys law 14.

1719 $x = (y\diamond y)\diamond((x\diamond x)\diamond y)$ has **unknown** spectrum containing $\{0, 1 \mod 3\} \cup \{5, 8\}$, possibly $[\![1, +\infty) \setminus \{2\}$? Indeed, it is obeyed by Mendelsohn quasigroups so the spectrum contains $\{0, 1 \mod 3\} \setminus \{6\}$, and magmas of size $5, 6, 8$ are easily found by an ATP.

## 2.3 Twists of the Dupont law 63

Next, we can discuss laws related to the Dupont law 63 and to law 73, its parastrophic equivalent.

63 $x = y\diamond(x\diamond(x\diamond y))$ (and 73, 118, 125, 1692) has **unknown** spectrum starting with $1, 3, 4, 5, 7, 8, 9, 11, 12, 13$ and containing most sizes $n \equiv 0, 1, 3 \mod 4$ up to 100. Many magmas (including some of size 9 that I think are non-linear) obey $x = y \diamond (y \diamond (y \diamond x))$ in addition to law 63.

Unipotent models ($x \diamond x = y \diamond y$) have sizes $1, 3, 9, \dots$ Idempotent models ($x \diamond x = x$) have sizes $1, 5, 7, 8, 11, \dots$ Models with a cyclicity property $x \diamond y = y \diamond z \iff y \diamond z = z \diamond x$ have sizes $1, 3, 4, 7, 9, 12, 13, \dots$

BLF A theorem of Richard M. Wilson https://doi.org/10.1016/0097-3165(75)90067-9

467 $x = y\diamond(x\diamond(x\diamond(y\diamond y)))$ has **unknown** spectrum starting with $1, 5, 7, 8$. It is the Dupont law twisted by the squaring map. The spectrum contains at least all odd integers that are sums of two squares $k^2 + l^2$ (with $k$, $l$ of different parities), as the quotient $\mathbb{Z}[i]/(k + il)\mathbb{Z}[i]$ equipped with the operation $x \diamond y = -\frac{k^2+l^2+1}{2}(1 + i)x + iy$ obeys this law. It has other linear models which are harder to describe. It likely has non-linear models too.

704 $x = y \diamond (y \diamond ((x \diamond x) \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is a twist of law 73 by the squaring map.

1110 $x = y\diamond((y\diamond(x\diamond x))\diamond y)$ has **unknown** spectrum starting with $1, 4, 5, 7, 8, 9$. It is a twist of law 73 by the squaring map.

1279 $x = y \diamond (((x \diamond x) \diamond y) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is (dual to) a twist of law 73 by the squaring map.

1516 $x = (y \diamond y) \diamond (x \diamond (x \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$. It is the Dupont law twisted by the squaring map.

## 2.4 Specializations of the central groupoid law 168

We know the spectrum of the central groupoid law 168 by a result of Knuth:

$$\mathrm{spec}(\mathrm{E168}) = \{n^2 \mid n \geq 1\}. \tag{10}$$

The spectrum of laws implied by law 168 thus contains $\{n^2 \mid n \geq 1\}$.

167 $x = (y \diamond x) \diamond (x \diamond y)$ has spectrum $\{0, 1 \mod 4\}$. Indeed, the map $f \colon (x, y) \mapsto (x \diamond y, y \diamond x)$, restricted to $x \neq y$, squares to the fixed-point-free involution $(x, y) \mapsto (y, x)$, so $f$ has orbits of size 4. As a result, the number of such ordered pairs must be $n(n-1) \equiv 0 \mod 4$, which forces $n \equiv 0, 1 \mod 4$. Constructing magmas of such sizes is straightforward using the same description. We give details in subsection 4.6 when counting the number of such magmas.

1480 $x = (y \diamond x) \diamond (x \diamond (x \diamond z))$ has **conjectural** spectrum $[\![1, +\infty) \setminus \{2, 3\}$, checked with an ATP up to order 18. Experimentally, it is even possible to take $x \diamond y = 0$ for $\lfloor \sqrt{n} \rfloor$ different values of $y$ (different from 0). There are models of all sizes $4 \cup [\![6, 11]\!]$ obeying law 1480 and law 166 $x = (y \diamond x) \diamond (x \diamond x)$, which are much easier to find for an ATP than just law 1480.

1483 $x = (y \diamond x) \diamond (x \diamond (y \diamond z))$ has **unknown** spectrum starting with $1, 2, 4, 8, 9$.

1485 $x = (y \diamond x) \diamond (x \diamond (z \diamond y))$ has **conjectural** spectrum $\{n^2 \mid n \geq 1\} \cup \{2n^2 \mid n \geq 1\}$ as discussed at https://leanprover.zulipchat.com/#narrow/stream/458659-Equational/topic/1485.

1486 $x = (y \diamond x) \diamond (x \diamond (z \diamond z))$ has **unknown** spectrum containing squares and $\{n^2 + 2 \mid n \geq 3\} \cup \{13, 21\}$, discussed at https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Understanding.20Finite.201486.20Magmas

## 2.5 Isolated laws

474 $x = y \diamond (x \diamond (y \diamond (x \diamond y)))$ has spectrum $[\![1, +\infty) \setminus \{2, 4\}$. It has models of all odd sizes $n = 2k + 1$ (which also obey law 1685), given by (for $0, x, y$ distinct) $0 \diamond 0 = 0$, $x \diamond x = 0$, $x \diamond 0 = x$, $0 \diamond x = \sigma(x)$ an involution without fixed point, and otherwise $x \diamond y = y$. It has a model of all even sizes $\geq 6$ given by an explicit multiplication table for $0 \leq x, y \leq 5$,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 4 | 5 | 1 |
| 1 | 1 | 0 | 5 | 4 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 | 5 | 4 |
| 3 | 3 | 5 | 4 | 0 | 2 | 1 |
| 4 | 4 | 2 | 1 | 5 | 0 | 3 |
| 5 | 5 | 4 | 3 | 2 | 1 | 0 |

(11)

together with $x \diamond x = 0$ and $x \diamond 0 = x$ for all $x$, and $0 \diamond x = \sigma(x)$ an involution without fixed point for $6 \leq x < n$, and $x \diamond y = y$ for all remaining entries, namely $\min(x, y) \geq 1$ and $\max(x, y) \geq 6$ and $x \neq y$.

670 $x = y \diamond (x \diamond ((x \diamond y) \diamond y))$ has **unknown** spectrum starting with $1, 4, 5$, not 6 nor 7

677 $x = y \diamond (x \diamond ((y \diamond x) \diamond y))$ has **unknown** spectrum. It is the source of the famous last surviving implication.

907 $x = y \diamond ((y \diamond x) \diamond (x \diamond y))$ has **unknown** spectrum containing $1, 3, 7, 9, 13$, and not $2, 4, 5, 6$. Models tend to either have a left identity or to be commutative and idempotent.

1076 $x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$ has **unknown** spectrum starting with $1, 5$, and not having 6 nor 7.

1286 $x = y \diamond (((x \diamond y) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 7$.

1313 $x = y \diamond (((y \diamond x) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7$.

1489 $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ has **conjectural** spectrum $[\![1, +\infty) \setminus \{2, 4\}$, see below.

1685 $x = (y \diamond x) \diamond ((x \diamond y) \diamond y)$ has spectrum $[\![1, +\infty) \setminus \{2\}$. It has models of all odd sizes $n = 2k + 1$ (which also obey law 474), given by (for $0, x, y$ distinct) $0 \diamond 0 = 0$, $x \diamond x = 0$, $x \diamond 0 = x$, $0 \diamond x = \sigma(x)$ an involution without fixed point, and otherwise $x \diamond y = y$. For even sizes $n = 2k + 4 \geq 4$, take $x \diamond y$ given by the following table

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 3 | 2 | 1 | 0 |
| 2 | 1 | 0 | 3 | 2 |
| 3 | 2 | 3 | 0 | 1 |

(12)

together with $x \diamond 0 = x$ and $x \diamond y = (y \diamond 0) \diamond 0$ for $1 \leq y \leq x$, and $x \diamond x = 0$ and $0 \diamond x = \sigma(x)$ for $4 \leq x$ (with $\sigma$ an involution of $[\![4, n-1]\!]$ without fixed points), and all other entries $x \diamond y = y$ (namely for $y \geq 4$ and $x \neq 0, y$).

### 2.5.1 Exploration of the equation 1489

Equation 1489 $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ has **conjectural** spectrum $[\![1, +\infty) \setminus \{2, 4\}$, checked up to order 21 included. **Curiously**, the smallest commutative models of law 1489 have size $1, 5, 21$ —we do not investigate further.

For all sizes except 5, there appears to be models obeying 1489 and law 4321 $x \diamond (y \diamond x) = y \diamond (x \diamond y)$. Together, these laws imply law 3 $x = x \diamond x$ and law 28 $x = (y \diamond x) \diamond x$ and law 1481 $x = (y \diamond x) \diamond (x \diamond (y \diamond x))$. In these models $x \Box y = x \diamond (y \diamond x)$ is commutative and obeys law 323 $x \Box y = x \Box (x \Box y)$, or equivalently law 333 $x \Box y = y \Box (x \Box y)$. One can define a graph with edges

$$(x \to y) \Leftrightarrow (x \Box y = y) \Leftrightarrow (y \Box x = y) \Leftrightarrow (\exists z, y = x \Box z) \Leftrightarrow (\exists z, y = z \Box x) \Leftrightarrow (x \diamond y = y). \tag{13}$$

Experimentally, the in and out degrees of each node are equal. Experimentally that degree (not counting self-loops) is between 3 and $\lfloor (n-1)/2 \rfloor$ included. For sizes $\{3, 6, 7, 8, 9\}$ at least, the graph can have cyclic symmetry, with degrees $\{1, 2, 3, 3, 3\}$, respectively. (However, the operation $\diamond$ itself, or even $\Box$, does not have cyclic symmetry, only approximate symmetry.) An example model on $[\![0, 8]\!]$ whose graph is cyclically symmetric has the operation $\diamond$ given by

```
[0, 3, 8, 3, 6, 2, 7, 7, 8],
[0, 1, 4, 0, 4, 7, 7, 8, 8],
[0, 1, 2, 5, 1, 5, 7, 8, 0],
[1, 1, 2, 3, 6, 2, 6, 8, 0],
[1, 2, 2, 3, 4, 7, 3, 7, 0],
[1, 2, 3, 3, 4, 5, 8, 4, 8],
[0, 3, 8, 4, 4, 5, 6, 0, 5],
[6, 1, 4, 0, 5, 5, 6, 7, 1],
[2, 7, 2, 5, 1, 6, 6, 7, 8]
```

and the operation $\Box$ given by

```
[[0, 0, 0, 3, 3, 3, 0, 7, 8],
 [0, 1, 1, 1, 4, 4, 0, 1, 8],
 [0, 1, 2, 2, 2, 5, 0, 1, 2],
 [3, 1, 2, 3, 3, 3, 6, 1, 2],
 [3, 4, 2, 3, 4, 4, 4, 7, 2],
 [3, 4, 5, 3, 4, 5, 5, 5, 8],
 [0, 0, 0, 6, 4, 5, 6, 6, 6],
 [7, 1, 1, 1, 7, 5, 6, 7, 7],
 [8, 8, 2, 2, 2, 8, 6, 7, 8]]
```

An example model on $[0, 16]$ without symmetry is given by

```
[0, 1, 1, 1, 4, 4, 3, 9, 1, 9,10,10, 2,13,13, 8,14,
 2, 1, 2, 3, 3, 2, 7, 7, 8, 0, 8, 2,15,12,11,15, 5,
 0, 0, 2, 1, 4, 5, 4, 5, 1,10,10,11,12,12, 8,11,14,
 5, 4, 6, 3, 4, 2, 6, 7, 7, 6, 0,11,11,13,11, 8,13,
 5, 1, 6, 1, 4, 5, 6, 1, 1, 9,10, 9,10,10,15,16,12,
 0, 6, 7, 8, 0, 5, 3, 7, 9, 9, 0,11, 6,11,10,16,16,
 7, 1, 2, 9, 2, 8, 6, 1, 1, 9,10, 2,12,10,15,15,12,
 0, 6, 2, 8, 2, 2, 6, 7, 8, 0, 0,11,14,10,11,11,12,
11,10, 9, 3,12, 5,13, 3, 8, 5,10, 2,14,15,14,15, 5,
```

```
 7, 6, 2, 3,11, 8, 3, 7, 8, 9, 2,11,11,12,11,11,13,
11, 1, 9,12,12,13,13,14, 1, 9,10,11,12,13,14, 1,14,
 0,12,15,12, 4,13,14,14,13, 4, 0,11,12,13,14,15, 5,
14, 1,13, 3, 4,15,16, 3, 8, 6, 4, 3,12,13, 8, 1,16,
14, 0, 2,16, 2, 5, 6, 1, 8,10, 6, 5, 2,13,14, 8,16,
 0, 6,13, 8, 2,15, 6, 7,12, 0,16, 7,12, 0,14, 6,16,
14,12, 2,16, 0, 5,14, 5,13,10, 4, 2,12,13,14,15, 5,
11,12, 9, 3, 2,15, 6, 1,13, 6,10, 2, 6, 3,10,15,16 ]
```

and the operation $x\Box y$ is given by

```
[0, 1, 0, 4, 4, 0, 9, 0,10, 9,10, 0,13,13, 0,13,10,
 1, 1, 2, 3, 1, 7, 1, 7, 8, 7, 1,15, 1, 2, 7,15,15,
 0, 2, 2, 4, 4, 5, 2, 2,10, 2,10,11,12, 2,12, 2,10,
 4, 3, 4, 3, 4, 7, 6, 7, 3, 3,11,11, 3,13, 7,13, 3,
 4, 1, 4, 4, 4, 5, 6, 6,10, 9,10, 4, 4, 6, 6, 5, 6,
 0, 7, 5, 7, 5, 5, 9, 7, 5, 9,11,11,16, 5,16, 5,16,
 9, 1, 2, 6, 6, 9, 6, 6,10, 9,10,15,12, 6, 6,15, 6,
 0, 7, 2, 7, 6, 7, 6, 7, 8, 7,11,11, 8, 6, 7, 2, 6,
10, 8,10, 3,10, 5,10, 8, 8, 8,10,15, 8, 8,14,15,15,
 9, 7, 2, 3, 9, 9, 9, 7, 8, 9, 9,11, 3, 2, 7, 2, 3,
10, 1,10,11,10,11,10,11,10, 9,10,11,12,13,14,12,10,
 0,15,11,11, 4,11,15,11,15,11,11,11,12,13,14,15,15,
13, 1,12, 3, 4,16,12, 8, 8, 3,12,12,12,13,12,12,16,
13, 2, 2,13, 6, 5, 6, 6, 8, 2,13,13,13,13,14,13,16,
 0, 7,12, 7, 6,16, 6, 7,14, 7,14,14,12,14,14,14,16,
13,15, 2,13, 5, 5,15, 2,15, 2,12,15,12,13,14,15,15,
10,15,10, 3, 6,16, 6, 6,15, 3,10,15,16,16,16,15,16]
```

and the (in or out) degrees are $(5,5,5,5,6,6,6,5,5,5,5,5,6,6,6,4,5)$ so this probably not the most symmetrical model possible.

## 3 Simple and (sub-)directly irreducible spectrum (todo)

We explain here in a streamlined way the results of the investigation discussed at https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Simple.20and.20.28sub.29directly.20irreducible.20spectrum/near/493052424

## 4 Counting magmas satisfying a law

Build enough semi-symmetric quasigroups (magmas obeying law 14) to saturate the upper bound even when the 2-adic valuation of $n$ is low, see subsubsection 4.2.3. Treat analogously law 66.

Ben Gunby-Mann started investigating the number of magmas satisfying low-lying laws in https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Equations.20with.20full.20spectrum/near/490261916 Continuing this investigation leads to asymptotic counts for most laws up to order 3. We determine here the number of magmas on a carrier set $M$ of cardinal $n$ subject to an equational law $A$. We consider two related numbers:

- $N_A(n)$ counting labeled magmas (namely counting magma operations on a fixed set $\{1, \dots, n\}$), and
- $I_A(n)$ counting isomorphism classes of magmas.

There are more general notions of equivalence (parastrophy etc.), but we do not consider them. For many laws $A$ one has $N_A(n) \sim n! I_A(n)$ because a typical magma has no symmetry. It is often easier to determine (or estimate) $N_A(n)$ and use the following obvious inequalities to control the asymptotics of $I_A(n)$,

$$\frac{1}{n!} N_A(n) \leq I_A(n) \leq N_A(n). \tag{14}$$

We summarize results in Table 4, Table 5, Table 6, Table 7, Table 8 according to the shape.

## 4.1 Different phases: vacuum, gas, liquid, solid, crystal

In our investigation we uncover several growth classes for the number of magmas obeying a law $A$, which we find analogous to "phases of matter".

$$\begin{aligned}
&\log N_A(n) = n^2 \log n + O(n \log n) && \text{vacuum,} \\
&\log N_A(n) = (1-p) n^2 \log n + o(n^2 \log n) && \text{gas with pressure } p \in [0,1), \\
&\log N_A(n) = n^{2-\rho+o(1)} && \text{liquid with density } \rho \in (0,1), \\
&\log N_A(n) \sim n \log n, \quad \log I_A(n) = o(n \log n) && \text{solid,} \\
&\log N_A(n) < n \log n, \quad \log I_A(n) = O(\log n) && \text{crystal.}
\end{aligned} \tag{15}$$

This list of phases is not meant to be a full classification of all possible behaviour for equational laws, and simply summarizes our findings so far. Vacuum laws typically allow for magmas where a fraction $O(1/n)$ of entries are quite rigid while the others are completely arbitrary, with the prototypical example being the idempotence law 3 ($x = x \diamond x$). Pressureless gases ($p = 0$) such as law 9 ($x = x \diamond (x \diamond y)$) are distinguished from vacuum ones by requiring that $n^2 \log n - \log N_A(n) \gg n \log n$. These laws typically have $\log N_A(n) = n^2(\log n - \log\log n + O(1))$, often arising as a maximum of $(n-m)^2 \log m$ for $0 \leq m \leq n$. See for instance the class of logarithmic projection magmas we define in (16). For other gas laws, the pressure $p$ measures what fraction of the operation table is constrained by the law, for instance law 16 ($x = y \diamond (y \diamond x)$) has $p = 1/2$ because a generic choice of $y \diamond x = u$ forces the value of $y \diamond u$ to be $x$. Liquid and solid laws are rare and the distinction may not be optimally chosen here.

An example of liquid law is the central groupoid law 168 ($x = (y \diamond x) \diamond (x \diamond z)$), whose rigid aspects implies that specifying $\sqrt{n}$ rows in the operation table is enough to characterize the magma. For solid laws the difference between $N_A(n)$ counting binary operations and $I_A(n)$ counting magmas up to isomorphism becomes important; a typical example is law 38 ($x \diamond x = x \diamond y$), which characterizes sets with an endo-function. Finally, crystal laws are such that their magmas always have symmetries (a magma isomorphism class without symmetry automatically leads to $N_A(n) \geq n!$ magma operations), for instance any model of law 13 ($x = y \diamond (x \diamond x)$), namely any set equipped with an involution, has symmetries permuting the involution's orbits. A few laws are anomalous, with an incomplete spectrum due to modular constraints, but otherwise fit into this classification of phases.

The remaining questions come in several varieties.

- The semi-symmetric law 14 where the lower bound is only proven for some families of sizes, but the asymptotics are likely available in the literature; likewise for law 66 the counting is closely related to Mendelsohn triples, which should be known.
- The two-variable Dupont law 63 (and the laws {73, 118, 125} which have the same spectrum) has enough idempotent models, leading us to some expectations for lower bounds of the asymptotic count based on general intuition from design theory (random graphs etc), but no proof yet. This can be achieved even through the spectrum of law 63 is not known.
- Some laws {115, 117, 162, 325, 335} are low-hanging fruit that seem approachable by understanding their structure better.
- Laws {49, 52, 53, 55, 58, 101, 102, 104, 105, 111, 152, 153, 309, 313, 315, 318, 319, 333} admit the lower bound $e^{n^2(\log n - \log\log n) + O(n^2)}$ (pressureless gas), but no upper bound is available yet. To prove that these asymptotics are also upper bounds, one has to show that the law constrains slightly all or almost all entries of the operation table, typically by forcing them to belong to some subset that cannot be too large.
- For some laws {65, 124} we have an ad-hoc construction of a large number of magmas based on involutions, but no clear upper bound.
- Finally, for law 168 we already have lower and upper bounds on $\log N_{168}(n)$ within constant factors of each other, and tightening this gap seems hard.

## 4.2 Some interesting magmas

### 4.2.1 Logarithmic projection magmas

BLF write a lemma about extremizing such functions. To get an upper bound, we consider the function $f(x) = \log(x^{n-x}) = (n-x)\log x$ for $x > 0$. One has $f'(x) = -\log x - 1 + n/x$ and $f''(x) = -1/x - n/x^2 < 0$ so it is concave,

Table 4: Number of magmas satisfying various laws up to order 2, sorted by their large-$n$ behaviour. Law 2 is not included here.

| Law $A$ | Equation | $\log N_A(n)$ | $\log I_A(n)$ | "Phase" |
|---|---|---|---|---|
| Law 1 | $x = x$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 3 | $x = x \diamond x$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 8 | $x = x \diamond (x \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 11 | $x = x \diamond (y \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 40 | $x \diamond x = y \diamond y$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 9 | $x = x \diamond (x \diamond y)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 10 | $x = x \diamond (y \diamond x)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 16 | $x = y \diamond (y \diamond x)$ | $(1/2)n^2 \log n + O(n^2)$ | | gas ($p = 1/2$) |
| Law 43 | $x \diamond y = y \diamond x$ | $(1/2)n^2 \log n + O(n \log n)$ | | gas ($p = 1/2$) |
| **Law 14** | $x = y \diamond (x \diamond y)$ | **likely** $(1/3)n^2 \log n + O(n^2)$ | | gas ($p = 2/3$) |
| Law 38 | $x \diamond x = x \diamond y$ | $n \log n$ | $\sim (1.08\dots)n$ | solid |
| Law 4 | $x = x \diamond y$ | 0 | 0 | crystal |
| Law 13 | $x = y \diamond (x \diamond x)$ | $\sim (1/2)n \log n$ | $\sim \log n$ | crystal |
| Law 41 | $x \diamond x = y \diamond z$ | $\log n$ | 0 | crystal |

Table 5: Number of magmas satisfying various laws of shape $\_ = \_\diamond(\_\diamond(\_\diamond\_))$, sorted by their large-$n$ behaviour.

| Law $A$ | Equation | $\log N_A(n)$ | $\log I_A(n)$ | "Phase" |
|---|---|---|---|---|
| Law 47 | $x = x \diamond (x \diamond (x \diamond x))$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 50 | $x = x \diamond (x \diamond (y \diamond y))$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 56 | $x = x \diamond (y \diamond (y \diamond y))$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 48 | $x = x \diamond (x \diamond (x \diamond y))$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 49 | $x = x \diamond (x \diamond (y \diamond x))$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 52 | $x = x \diamond (y \diamond (x \diamond x))$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 53 | $x = x \diamond (y \diamond (x \diamond y))$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 55 | $x = x \diamond (y \diamond (y \diamond x))$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 58 | $x = x \diamond (y \diamond (z \diamond x))$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 75 | $x = y \diamond (y \diamond (y \diamond x))$ | $(2/3)n^2 \log n + O(n^2)$ | | gas ($p = 1/3$) |
| **Law 65** | $x = y \diamond (x \diamond (y \diamond x))$ | $\geq (1/2)n^2 \log n + O(n^2 \log\log n)$ | | gas ($\boldsymbol{p \leq 1/2}$) |
| Law 72 | $x = y \diamond (y \diamond (x \diamond x))$ | $(1/2)n^2 \log n + O(n^2)$ | | gas ($p = 1/2$) |
| **Law 66** | $x = y \diamond (x \diamond (y \diamond y))$ | $n \equiv 0, 1 \mod 3$ | | anomalous |
| **Law 63** | $x = y \diamond (x \diamond (x \diamond y))$ | **unknown spectrum** | | |
| Law 73 | $x = y \diamond (y \diamond (x \diamond y))$ | same as law 63 | | |
| Law 62 | $x = y \diamond (x \diamond (x \diamond x))$ | $\sim (2/3)n \log n$ | $\sim \log n$ | crystal |

Table 6: Number of magmas satisfying various laws of shape _ = _◇((_◇_)◇_), sorted by their large-$n$ behaviour.

| Law $A$ | Equation | $\log N_A(n)$ | $\log I_A(n)$ | "Phase" |
|---|---|---|---|---|
| Law 99 | $x = x \diamond ((x \diamond x) \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 100 | $x = x \diamond ((x \diamond x) \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 107 | $x = x \diamond ((y \diamond y) \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 108 | $x = x \diamond ((y \diamond y) \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 109 | $x = x \diamond ((y \diamond y) \diamond z)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 101 | $x = x \diamond ((x \diamond y) \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 102 | $x = x \diamond ((x \diamond y) \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 103 | $x = x \diamond ((x \diamond y) \diamond z)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 104 | $x = x \diamond ((y \diamond x) \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 105 | $x = x \diamond ((y \diamond x) \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 106 | $x = x \diamond ((y \diamond x) \diamond z)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 111 | $x = x \diamond ((y \diamond z) \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| **Law 124** | $x = y \diamond ((y \diamond x) \diamond x)$ | $\geq (1/2)n^2 \log n + O(n^2 \log\log n)$ | | gas ($\boldsymbol{p \leq 1/2}$) |
| Law 127 | $x = y \diamond ((y \diamond y) \diamond x)$ | $(1/2)n^2(\log n - 1) + O(n^{3/2})$ | | gas ($p = 1/2$) |
| **Law 117** | $x = y \diamond ((x \diamond y) \diamond x)$ | $\leq (1/2)n^2 \log n + O(n^2)$ | | gas ($\boldsymbol{p \geq 1/2}$) |
| Law 138 | $x = y \diamond ((z \diamond y) \diamond x)$ | $(1/4)n^2 \log n + O(n^2)$ | | gas ($p = 3/4$) |
| **Law 115** | $x = y \diamond ((x \diamond x) \diamond y)$ | | | |
| Law 118 | $x = y \diamond ((x \diamond y) \diamond y)$ | | same as law 63 | |
| Law 125 | $x = y \diamond ((y \diamond x) \diamond y)$ | | same as law 63 | |

Table 7: Number of magmas satisfying various laws of shape _ = (_◇_)◇(_◇_), sorted by their large-$n$ behaviour.

| Law $A$ | Equation | $\log N_A(n)$ | $\log I_A(n)$ | "Phase" |
|---|---|---|---|---|
| Law 151 | $x = (x \diamond x) \diamond (x \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 152 | $x = (x \diamond x) \diamond (x \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 153 | $x = (x \diamond x) \diamond (y \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 167 | $x = (y \diamond x) \diamond (x \diamond y)$ | $(1/2)n^2 \log n + O(n^2)$ | | gas ($p = 1/2$) |
| Law 159 | $x = (x \diamond y) \diamond (y \diamond x)$ | $(1/2)n^2 \log n + O(n^2)$ | | gas ($p = 1/2$) |
| **Law 162** | $x = (x \diamond y) \diamond (z \diamond x)$ | | | |
| **Law 168** | $x = (y \diamond x) \diamond (x \diamond z)$ | $\sim \mathbf{[1/8, 1/2]} n^{3/2} \log n$ | | liquid ($\rho = 1/2$) |

Table 8: Number of magmas satisfying various laws of shape $\_\diamond\_ = \_\diamond(\_\diamond\_)$, sorted by their large-$n$ behaviour.

| Law $A$ | Equation | $\log N_A(n)$ | $\log I_A(n)$ | "Phase" |
|---|---|---|---|---|
| Law 307 | $x \diamond x = x \diamond (x \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 310 | $x \diamond x = x \diamond (y \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 312 | $x \diamond x = y \diamond (x \diamond x)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 316 | $x \diamond x = y \diamond (y \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 326 | $x \diamond y = x \diamond (y \diamond y)$ | $n^2 \log n + O(n \log n)$ | | vacuum |
| Law 308 | $x \diamond x = x \diamond (x \diamond y)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 309 | $x \diamond x = x \diamond (y \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 311 | $x \diamond x = x \diamond (y \diamond z)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 313 | $x \diamond x = y \diamond (x \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 314 | $x \diamond x = y \diamond (x \diamond z)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 315 | $x \diamond x = y \diamond (y \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 317 | $x \diamond x = y \diamond (y \diamond z)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 318 | $x \diamond x = y \diamond (z \diamond x)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 319 | $x \diamond x = y \diamond (z \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 323 | $x \diamond y = x \diamond (x \diamond y)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 333 | $x \diamond y = y \diamond (x \diamond y)$ | $\geq n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 343 | $x \diamond y = z \diamond (x \diamond y)$ | $n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 0$) |
| Law 332 | $x \diamond y = y \diamond (x \diamond x)$ | $(1/2)n^2(\log n - \log\log n) + O(n^2)$ | | gas ($p = 1/2$) |
| **Law 325** | $x \diamond y = x \diamond (y \diamond x)$ | | | |
| Law 327 | $x \diamond y = x \diamond (y \diamond z)$ | $(1/4)n^2 \log n + O(n^2)$ | | gas ($p = 3/4$) |
| **Law 335** | $x \diamond y = y \diamond (y \diamond x)$ | | | |
| Law 329 | $x \diamond y = x \diamond (z \diamond y)$ | $((\log 3)/3)n^2 + O(n \log n)$ | | liquid ($\rho = 0$) |
| Law 336 | $x \diamond y = y \diamond (y \diamond y)$ | $\sim n \log n$ | $\sim \pi\sqrt{2n/3}$ | crystal |

with a maximum at $x \log x + x = n$, namely (for large $n$) $x = n(\log n)^{-1} + O(n(\log n)^{-2} \log\log n)$, from which we find $f(x) \leq n \log n - n \log\log n + O(n)$.

We introduce the class of "logarithmic projection magmas" on $[\![0, n-1]\!]$ as follows. Set $m = \lfloor n/\log n \rfloor$. Set all $i \diamond i = i$. For $i, j \in [\![0, n-1]\!]$ with $\min(i, j) < m$ set $i \diamond j = i$. Set the remaining $(n-m)(n-m-1)$ entries to have values in $[\![0, m-1]\!]$:

$$\begin{aligned} &i \diamond j = i \quad \text{if } i < m \text{ or } j < m \text{ or } i = j, \\ &i \diamond j \in [\![0, m-1]\!] \quad \text{otherwise.} \end{aligned} \tag{16}$$

The number of such magmas is

$$N_{\log\,\mathrm{proj}}(n) = m^{(n-m)(n-m-1)} = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{17}$$

As we now prove, these magmas satisfy many laws: 9, 10, 48, 49, 50, 52, 100, 101, 102, 103, 104, 105, 106, 107, 152, 153, 205, 208, 308, 309, 326, 375, 377, 378, as well as all single-variable laws since they are idempotent magmas. All of these laws $A$ that holds in these magmas are thus (at least) pressureless gas laws since $N_A(n) \geq N_{\log\,\mathrm{proj}}(n)$. Let us check the equations in turn. For $x < m$ they are checked immediately, so we focus on $x \geq m$.

9 $x = x \diamond (x \diamond y)$: one has $x \diamond y \in [\![0, m-1]\!] \cup \{x\}$ and left multiplying again by $x$ gives $x$.

10 $x = x \diamond (y \diamond x)$: if $x \geq m$ and $y \in [\![0, m-1]\!] \cup \{x\}$ then $y \diamond x = y$ and $x \diamond y = x$; otherwise $y \diamond x \in [0, m)$ and left-projection also applies.

103 $x = x \diamond ((x \diamond y) \diamond z)$: one has $(x \diamond y) \diamond z \in \{x \diamond y, x \diamond z\}$ since either $x \diamond y = x$ or $x \diamond y < m$ and the operation reduces to left projection; thus $x \diamond ((x \diamond y) \diamond z) = x \diamond (x \diamond w)$ for $w \in \{y, z\}$ and we conclude by law 9.

106 $x = x \diamond ((y \diamond x) \diamond z)$: for $x \geq m$ and $y = x$ law 9 and idempotence conclude; for $x \geq m$ and $y \in [\![0, m-1]\!]$ we have $(y \diamond x) \diamond z = y \diamond z = y$ and $x \diamond y = x$; and otherwise $y \diamond x < m$ so $(y \diamond x) \diamond z = y \diamond x$ and law 10 applies.

377 $x \diamond y = (x \diamond y) \diamond x$: for $x \geq m$ and $y \in [\![0, m-1]\!] \cup \{x\}$ we have $x \diamond y = x$ and idempotence concludes; otherwise $x \diamond y < m$ and left projection applies.

378 $x \diamond y = (x \diamond y) \diamond y$: for $x \geq m$ and $y \in [\![0, m-1]\!] \cup \{x\}$ we have $x \diamond y = x$; otherwise $x \diamond y < m$ and left projection applies.

Additionally, laws 48, 49, 50, 100, 152, 205, 308 are consequences of law 9 and idempotence, laws 52, 101, 104, 107, 153, 208, 309 are consequences of law 10 and idempotence, and laws 102, 105, 326, 327 are consequences of laws 103, 106, idempotence, and idempotence, respectively.

#### 4.2.2 Lower bound on the number of quasigroups

It is known that the number $N_{\mathrm{Latin}}(n) = $ A002860$(n)$ of Latin squares (labelled quasigroups) is bounded below as

$$N_{\mathrm{Latin}}(n) \geq \left(n!^2/n^n\right)^n = e^{n^2(\log n - 2) + O(n \log n)}. \tag{18}$$

(An intuition for this result is that a random map $[\![0, n-1]\!] \to [\![0, n-1]\!]$ has probability roughly $e^{-n}$ to be a permutation (since $n!/n^n = e^{-n+O(\log n)}$) and we want $2n$ such conditions, for each left and right multiplications, which explains the $e^{-2n^2}$ factor.)

proof sketch at https://math.stackexchange.com/questions/4166583/explanation-for-the-fracn2nnn2-lower-bound-on-the-number-of-lat

#### 4.2.3 Construction of semi-symmetric quasigroups

Next, we consider magmas satisfying the semi-symmetric quasigroup law 14 $x = y \diamond (x \diamond y)$. Our aim is to show that $\log N_{14}(n) \sim (1/3)n^2 \log n$. We first determine an upper bound. The magma operations can be enumerated by the following procedure. First we have to choose the squaring map $S\colon M \to M$. Then, consider the list $L$ of pairs $(x, y)$ with $x \neq y$ and $x \neq S(y)$ and $y \neq S(x)$, and sort $L$ in lexicographic order (say). For the first pair in that list $L$, decide the product $x \diamond y \in M \setminus \{x, y\}$; it must be such that the pairs $(y, x \diamond y)$ and $(x \diamond y, x)$ are in the list $L$. Delete these two pairs from the list $L$. Then move on to choosing the product for the next pair in $L$, and so on. This process either exhausts $L$ or gets stuck at a point where there is no valid choice of $x \diamond y$. Every step reduces $|L|$ by 3, so the process stops in at most $(n^2 - n)/3$ steps, and we get

$$N_{14}(n) \leq n^n (n-2)^{(n^2-n)/3} = e^{(1/3)n^2 \log n + O(n^2)}. \tag{19}$$

Known numerical values of A076016$(n)$ are not sufficient to test whether this coefficient is indeed correct. We show that this asymptotics is **correct for $n = 2^k m$ with $k \to +\infty$ and fixed** $m \geq 1$, as follows.

Given a quasigroup $(Q, \cdot)$ of cardinality $n/2$ and a semi-symmetric quasigroup with the same underlying set $(Q, \circ)$ (namely $\circ$ obeys law 14), we define $M = Q \times \{0, 1\}$ with the operation

$$\begin{aligned} (x, 0) \diamond (y, 0) &= (x \cdot y, 1), \\ (y, 0) \diamond (x \cdot y, 1) &= (x, 0), \\ (x \cdot y, 1) \diamond (x, 0) &= (y, 0), \\ (x, 1) \diamond (y, 1) &= (x \circ y, 1). \end{aligned} \tag{20}$$

The second and third identities here define fully the products $(\_, 0) \diamond (\_, 1)$ and $(\_, 1) \diamond (\_, 0)$ because $(Q, \cdot)$ is a quasigroup. This operation obeys law 14, thus

$$N_{14}(n) \geq N_{14}(n/2) N_{\text{Latin}}(n/2) = N_{14}(n/2) e^{(1/4)n^2(\log n - 2 - \log 2) + O(n \log n)}. \tag{21}$$

Iterating $k$ times this identity (for $n = 2^k m$), and using $N_{14}(m) \geq 1$ (thanks to the explicit model $\mathbb{Z}/m\mathbb{Z}$ with $x \circ y = -x - y \mod m$),

$$\log N_{14}(n) \geq \frac{1 - 2^{-2v_2(n)}}{3} n^2 \log n - \Big(\frac{2}{3} + \frac{4}{9} \log 2\Big) n^2 + O(n (\log n)^2), \tag{22}$$

where $k = v_2(n)$ is the 2-adic valuation of $n$. For $n$ in the sequence $\{2^k m \mid k \geq 1\}$, the first term behaves as $(1/3)n^2 \log n + O(\log n)$. In fact, **if one could**

**prove that $N_{14}(n)$ is monotonically increasing,** then the information we have would be enough to deduce $\log N_{14}(n) \sim (1/3)n^2 \log n$. Alternatively, one could try to **tweak the construction above** to allow for an additional element: instead of extending a semi-symmetric quasigroup by a quasigroup, we could maybe extend a semi-symmetric loop by a loop and decide whether to identify the two identity elements or not depending on parity of $n$?

NB: Law 887 characterizes semi-symmetric loops, namely magmas satisfying both law 14 and the unipotence law 40 $x \diamond x = y \diamond y$ (the common value of the squares is then a two-sided identity $e$). Such magmas are in one-to-one correspondence with Mendelsohn triple systems on the underlying set with $e$ removed. Likewise, law 4961 characterizes idempotent semi-symmetric quasigroups, namely magmas satisfying both law 14 and the idempotence law 3 $x \diamond x = x$, which are in one-to-one correspondence with Mendelsohn triple systems on the set itself. We have $I_{887}(m) = I_{4961}(m-1) =$ A076021$(m-1)$.

#### 4.2.4 Construction of central groupoids

Knuth gave a one-to-one correspondence between such magmas and directed graphs where $x \to y$ if and only if $y \in x \diamond M$ if and only if $x \in M \diamond y$. He showed that the cosets $x \diamond M$ and $M \diamond x$ all have $m$ elements. We immediately get an upper bound on $N_{168}(n)$ by counting all possible choices of $n$ cosets $x \diamond M$, namely

$$N_{168}(n) \leq \binom{n}{m}^n \leq e^{n^{3/2} \log n}. \tag{23}$$

This upper bound can also be seen from https://doi.org/10.1016/j.jcta.2003.10.001 Theorem 2.5 which expresses the adjacency matrix of the central groupoid as a sum of $m$ permutation matrices, so $N_{168}(n) \leq \binom{n!}{m} \leq n!^m \leq n^{n^{3/2}}$.

We now gain a factor of $1/2$ in the exponential by better organizing the information. For each $x$ the $m$ cosets $\{y \diamond M \mid y \in x \diamond M\}$ form a partition of the magma into $m$-element subsets. The number of such (labeled) partitions is $n!/m!^m = e^{(1/2)n \log n + O(n^{1/2})}$. The magma is characterized as follows starting from any reference element 0: pick its coset $0 \diamond M$ and corresponding partition $\{x \diamond M \mid x \in 0 \diamond M\}$; then for each $x \in 0 \diamond M$ (whose coset has already been chosen), pick the partition $\{y \diamond M \mid y \in x \diamond M\}$, with the constraint (which we do not account for when counting) that some cosets have already been chosen in previous steps. Thus,

$$N_{168} \leq \binom{n}{m} \left(\frac{n!}{m!^m}\right)^{1+m} = e^{(1/2)n^{3/2} \log n + O(n \log n)}. \tag{24}$$

Conversely, Knuth constructed a class of central groupoids based on choosing a left quasigroup with $m$ elements that includes an element 0 that is a left identity and right zero. There are only $e^{(1/2)n \log n + O(n)}$ such magmas, but we can generalize the construction. Denote $l \in (0, m)$ for $l$ a parameter chosen momentarily (Knuth's construction corresponds to $l = 1$), and consider $m^2$ permutations $\sigma_{ab}$ of $[\![0, m-1]\!]$ labeled by $a, b \in [\![0, m-1]\!]$, such that $\sigma_{ab}(c) = c$

whenever $a \geq l$ or $c < l$. In particular $\sigma_{ab}(c) \in [\![l, m-1]\!]$ for $c \in [\![l, m-1]\!]$. Define a binary operation $\diamond$ on $[\![0, m-1]\!]^2$ by (for $a, b, c, d \in [\![0, m-1]\!]$),

$$(a,b) \diamond (c,d) = \bigl(\sigma_{ca}(b), (\sigma_{db})^{-1}(c)\bigr). \tag{25}$$

Then we check that law 168 is satisfied,

$$\begin{aligned}
&\bigl((c,d) \diamond (a,b)\bigr) \diamond \bigl((a,b) \diamond (e,f)\bigr) \\
&= \bigl(\sigma_{ac}(d), \sigma_{bd}^{-1}(a)\bigr) \diamond \bigl(\sigma_{ea}(b), \sigma_{fb}^{-1}(e)\bigr) \\
&= \bigl(\sigma_{\sigma_{ea}(b)\sigma_{ac}(d)}\sigma_{bd}^{-1}(a),\ \sigma^{-1}_{\sigma_{fb}^{-1}(e)\sigma_{bd}^{-1}(a)}\sigma_{ea}(b)\bigr).
\end{aligned} \tag{26}$$

We would like this to be equal to $(a,b)$. Let us focus on proving that the first entry is $a$. For $a \in [\![0, l-1]\!]$ this is immediate since all permutations fix $[\![0, l-1]\!]$. For $a \in [\![l, m-1]\!]$, we have $\sigma_{ac}(d) = d$. Distinguish $b \in [\![0, l-1]\!]$ for which $\sigma_{ea}(b) = b$ and we have $\sigma_{bd}\sigma_{bd}^{-1}(a) = a$, and $b \in [\![l, m-1]\!]$ for which $\sigma_{ea}(b) \in [\![l, m-1]\!]$ and the two permutations $\sigma_{\sigma_{ea}(b)\dots}$ and $\sigma_{bd}$ are equal to the identity. The second entry is treated similarly. We can now count how many magmas we have built: accounting for the fixed values of some $\sigma_{ab}(c)$, we have $lm$ permutations of $(m-l)$. To (almost) maximize the lower bound we take $l = m/2$,

$$\begin{aligned}
N_{168}(n) &\geq \max_{l \in (0,m)} (m-l)!^{lm} = e^{lm(m-l)(\log(m-l)-1)+O(lm)} \\
&\geq e^{(1/8)n^{3/2}(\log n - 2\log 2 - 2)+O(n)}.
\end{aligned} \tag{27}$$

We have thus found that $(\log N_{168}(n))/(n^{3/2}\log n)$ is asymptotically bounded in $[1/8, 1/2]$, but have not determined the precise constant.

## 4.3 Order up to 2

In this discussion we consider in turn one law in each equivalence class, omitting laws whose dual is (equivalent to) a lower-numbered law. Note that if law $A$ implies law $B$ then $N_B(n) \leq N_A(n)$ and $I_B(n) \leq I_A(n)$. In particular, law 3 (which has a lot of models) implies all single-variable laws.

- Law 1 ($x = x$): the magma operation is unconstrained and consists of choosing $n^2$ times among $n$ choices so $N_1(n) = n^{n^2}$.
- Law 2 ($x = y$): only possible for $n = 1$, so $N_2(n) = I_2(n) = \delta_{n=1}$.
- Law 3 ($x = x \diamond x$): diagonal entries of the Cayley table are fixed, the others are arbitrary so $N_3(n) = n^{n^2 - n}$.
- Law 4 ($x = x \diamond y$): the law fully fixes the operation so $N_4(n) = I_4(n) = 1$.
- Law 8 ($x = x \diamond (x \diamond x)$): each left-multiplication is constrained independently, either $L_x(x) = x$ and the $n-1$ remaining $L_x(y)$ are arbitrary, or $L_x$ maps $x$ to another value itself mapped to $x$. Thus $N_8(n) = ((2n-1)n^{n-2})^n > N_3(n)$.

- Law 9 ($x = x \diamond (x \diamond y)$): each left-multiplication $L_x$ is constrained independently, and this law implies idempotence. The choice of $L_x$ amounts to the choice of subset $L_x^{-1}(x) \setminus \{x\}$ and of map $(M \setminus L_x^{-1}(x)) \to L_x^{-1}(x) \setminus \{x\}$, so

$$N_9(n) = \Big(\sum_{k=1}^{n-1} \binom{n-1}{k} k^{n-1-k}\Big)^n = (\text{A000248}(n-1))^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{28}$$

  The logarithmic projection magmas (16) correspond to choosing all $L_x^{-1}(x)$ equal to the same subset (except for the element $\{x\}$) with $\lfloor n/\log n \rfloor$ elements and account for the same growth rate.

- Law 10 ($x = x \diamond (y \diamond x)$): such a magma is characterized by an unoriented graph (which can have loops) and for every $x \in M$ a map from non-neighbors of $x$ to neighbors of $x$, so there are (denoting by $\deg_E(i)$ the number of edges $(i, \dots)$ or $(\dots, i)$ in $E$)

$$\sum_{E \subset \{(i,j), 1 \le i \le j \le n\}} \prod_{i=1}^{n} \big(\deg_E(i)\big)^{n - \deg_E(i)}. \tag{29}$$

  A lower bound $N_{10}(n) \ge e^{n^2(\log n - \log\log n + O(1))}$ is provided by logarithmic projection magmas (16), corresponding to a graph with $\deg_E(i) \simeq n/\log n$ for all $i$. We can see this lower bound in the present language by noting that the term for this graph scales as $(n/\log n)^{n^2(1-1/\log n)} = e^{n^2(\log n - \log\log n + O(1))}$. Conversely, every term in the sum is bounded by $\max_m m^{n(n-m)} = e^{n^2(\log n - \log\log n + O(1))}$, and there are $2^{n(n+1)/2} = e^{O(n^2)}$ graphs, which leads to the asymptotics $N_{10}(n) = e^{n^2(\log n - \log\log n + O(1))}$.

- Law 11 ($x = x \diamond (y \diamond y)$): such magmas are characterized by the set $Q$ of squares, a squaring map $M \setminus Q \to Q$ (for $x \in Q$, $x \diamond x = x$), and the remaining entries $x \diamond y$ for $y \in M \setminus Q$ and $x \in M \setminus \{y\}$, so

$$N_{11}(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k} n^{(n-1)(n-k)} = n^{n^2 - 2n + O(1)}, \tag{30}$$

  with most of these being accounted for by unipotent magmas (all squares equal, $k = 1$).

- Law 13 ($x = y \diamond (x \diamond x)$): the operation is right-projection composed with an involution, so

$$N_{13}(n) = \text{A000085}(n) = e^{(1/2)n(\log n - 1) + \sqrt{n} + O(1)}. \tag{31}$$

  Modulo isomorphism, only the number of 2-cycles matters, which ranges from 0 to $\lfloor n/2 \rfloor$, so $I_{13}(n) = \lfloor n/2 \rfloor + 1$.

- Law 14 ($x = y \diamond (x \diamond y)$): $N_{14}(n) =$ A076016$(n)$ and $I_{14}(n) =$ A076017$(n)$. The asymptotics are difficult to establish so we have studied them separately in subsubsection 4.2.3.
- Law 16 ($x = y \diamond (y \diamond x)$): each left-multiplication is constrained independently to be an involution, so $N_{16}(n) = N_{13}(n)^n =$ A000085$(n)^n$. See (31) for asymptotics.
- Law 38 ($x \diamond x = x \diamond y$): the operation is left-projection composed with squaring, which is an arbitrary endofunction, so $N_{38}(n) = n^n$, and $I_{38}(n) = A001372(n) \sim C_2\, C_1^n/\sqrt{n}$ for some constants $C_1 \simeq 2.96$ and $C_2 \simeq 0.443$, see `https://oeis.org/A001372`.
- Law 40 ($x \diamond x = y \diamond y$): magmas in which all elements square to the same element (which we have to pick) so $N_{40}(n) = n^{n^2-n+1}$.
- Law 41 ($x \diamond x = y \diamond z$): all products are equal to the same element, so $N_{41}(n) = n$, and all choices are isomorphic so $I_{41}(n) = 1$.
- Law 43 ($x \diamond y = y \diamond x$): commutativity means we only pick $n(n+1)/2$ entries in the Cayley table, so $N_{43}(n) = n^{n(n+1)/2}$.

### 4.4 Order 3: first shape $\_ = \_ \diamond (\_ \diamond (\_ \diamond \_))$

- Law 47 ($x = x \diamond (x \diamond (x \diamond x))$) is a single-variable law hence $N_{47}(n) \geq N_3(n) = n^{n^2-n}$. In detail, each left multiplication is chosen separately, either $x \diamond x = x$ and other entries are unspecified, or one has to choose distinct $x \diamond x, x \diamond (x \diamond x)$, so $N_{47}(n) = ((2n^2 - 3n + 2)n^{n-3})^n$.
- Law 48 ($x = x \diamond (x \diamond (x \diamond y))$) is similar to law 9; it constrains each left multiplication $L_x$ independently to cube to the constant $x$ (and in particular $L_x(x) = x$). Such maps are in one-to-one correspondence with maps $f\colon M \setminus \{x\} \to M \setminus \{x\}$ with $f(f(y)) = f(f(f(y)))$, by assigning $L_x(y) = f(y)$ if that is different from $y$ and otherwise $L_x(y) = x$. Thus $N_{48}(n) =$ A000949$(n-1)^n$, whose asymptotics are known. Since law 48 holds in logarithmic projection magmas (16) one knows $N_{48}(n) \geq e^{n^2(\log n - \log\log n) + O(n^2)}$.
- Law 49 ($x = x \diamond (x \diamond (y \diamond x))$) holds in logarithmic projection magmas (16) hence $N_{49}(n) \geq e^{n^2(\log n - \log\log n + O(1))}$. This is **probably** also the correct asymptotics, but one would have to prove an upper bound. We dub it "gas ($p = 0$)" even though it could be closer to a "vacuum".
- Law 50 ($x = x \diamond (x \diamond (y \diamond y))$) is a consequence of law 9, so $N_{50}(n) \geq e^{n^2(\log n - \log\log n + O(1))}$. We can do better. On $\{1, \dots, n\}$ with $n \geq 3$, set all $x \diamond x = 1$ so that the law reduces to a condition $L_x(L_x(1)) = x$ for all $x \neq 1$. This is easily achieved by taking $L_x(1) \in M \setminus \{1, x\}$ and then $L_x$ of that to be $x$. The remaining $n - 3$ entries of $L_x$ are arbitrary. Thus

$$N_{50}(n) \geq (n-2)^n n^{n(n-3)} = e^{n^2 \log n + O(n \log n)}. \tag{32}$$

- Law 52 ($x = x \diamond (y \diamond (x \diamond x))$) holds in logarithmic projection magmas (16) hence $N_{52}(n) \geq e^{n^2(\log n - \log\log n + O(1))}$. The reverse inequality is unclear, same comments as for law 49 apply.

- Law 53 ($x = x \diamond (y \diamond (x \diamond y))$): one can construct many magmas of size $n$ as follows. Pick an idempotent semi-symmetric quasigroup $(Q, \cdot)$ (namely a magma satisfying laws 3 and 14, such as $\mathbb{Z}/3\mathbb{Z}$ equipped with $x \cdot y = -x - y$), and a surjective map $\pi\colon M \to Q$ and take

$$\begin{aligned} &x \diamond y = x && \text{if } \pi(x) = \pi(y), \\ &x \diamond y \in \pi^{-1}(\pi(x) \cdot \pi(y)) && \text{otherwise.} \end{aligned} \tag{33}$$

  Note that $\pi(x \diamond y) = \pi(x) \cdot \pi(y)$ in all cases since $(Q, \cdot)$ is idempotent. Since $\pi(y \diamond (x \diamond y)) = \pi(y) \cdot (\pi(x) \cdot \pi(y)) = \pi(x)$ the first rule applies when computing $x \diamond (y \diamond (x \diamond y))$, which is thus equal to $x$ as wanted. This construction is very flexible. Denoting $q = |Q|$, there are at least $\lfloor n/q \rfloor^{q(q-1)\lfloor n/q \rfloor^2}$ such magmas. At large $n$, taking $q = \log n$, we obtain $e^{n^2(\log n - \log\log n) + O(n^2)}$ magmas.

- Law 55 ($x = x \diamond (y \diamond (y \diamond x))$): implied by law 58, below, hence $N_{55}(n) \geq N_{58}(n)$.

- Law 58 ($x = x \diamond (y \diamond (z \diamond x))$): for some surjection $\pi\colon M \to Q$ for some set $Q$, consider

$$\begin{aligned} &x \diamond y = x && \text{if } \pi(x) = \pi(y), \\ &x \diamond y \in \pi^{-1}(y) && \text{if } \pi(x) \neq \pi(y). \end{aligned} \tag{34}$$

  In all cases $\pi(x \diamond y) = \pi(y)$, so $\pi(y \diamond (z \diamond x)) = \pi(x)$, so when computing $x \diamond (y \diamond (z \diamond x))$ the first case applies and we get $x$ as desired. For a fixed set $Q$ of cardinality $q$, we can take the sets $\pi^{-1}(i)$ to each have at least $\lfloor n/q \rfloor$ elements, and we find $N_{58}(n) \geq \lfloor n/q \rfloor^{q(q-1)\lfloor n/q \rfloor^2}$. Taking $q = \log n$ as for law 53 we get the announced asymptotics.

- Law 56 ($x = x \diamond (y \diamond (y \diamond y))$): a class of models is given by $0 \diamond 0 = 0$, $1 \diamond 0 = 1$, $1 \diamond 1 = 2$, $1 \diamond 2 = 0$, and, for $x \geq 2$, $x \diamond 0 = x$, $x \diamond 1 = 0$ and $x \diamond x = 1$. Indeed, these identities imply $y \diamond (y \diamond y) = 0$ and $x \diamond 0 = x$ for all $x, y$. There remains $n(n-2)$ entries to specify arbitrarily, so $N_{56}(n) \geq n^{n(n-2)}$.

- Law 62 ($x = y \diamond (x \diamond (x \diamond x))$): the magma operation is right-projection composed with a permutation $\sigma$ with $\sigma^{\circ 3} = \mathrm{id}$, so $N_{62}(n) = $ A001470$(n)$. Up to isomorphism, only the number of 3-cycles matters, so $I_{62}(n) = \lfloor n/3 \rfloor + 1$.

- Law 65 ($x = y \diamond (x \diamond (y \diamond x))$): a class of such magmas is build as follows. Fix a surjection $\pi\colon M \to Q$ to some set. For each $x \in M$ and $i \in Q$ with $\pi(x) \neq i$, select an involution $f_{x,i}\colon \pi^{-1}(i) \to \pi^{-1}(i)$, and for $\pi(x) = i$ set $f_{x,i}$ to be the identity. Then define $x \diamond y = f_{x,\pi(y)}(y)$. By construction $\pi(x \diamond y) = \pi(y)$ so $x$ and $y \diamond x$ have the same image under $\pi$ and hence

$x \diamond (y \diamond x) = y \diamond x$. Thus, $y \diamond (x \diamond (y \diamond x)) = y \diamond (y \diamond x) = f^{\circ 2}_{y,\pi(x)}(x) = x$. The number of magmas built this way, for a fixed $q$ and surjection whose preimages each have at least $\lfloor n/q \rfloor$ elements, is a power of the number $N_{13}$ of involutions,

$$N_{65}(n) \geq \left(N_{13}(\lfloor n/q \rfloor)\right)^{n(q-1)} \overset{q=\log n}{=} e^{(1/2)n^2(\log n - \log\log n)+O(n^2)}. \tag{35}$$

- Law 66 ($x = y \diamond (x \diamond (y \diamond y))$): such magmas are in one-to-one correspondence with Mendelsohn quasigroups (law 4961, equivalently idempotence law 3 and semi-symmetry law 14) equipped with an involutive automorphism. The construction of law 14 magmas **can probably** be restricted to idempotent magmas without changing the leading asymptotics. The involutive automorphism changes the number of such magmas by a factor between 1 (it can be the identity) and $n!$ (it has to be a bijection), which also leaves the leading asymptotics unchanged.
- Law 72 ($x = y \diamond (y \diamond (x \diamond x))$): squaring is a permutation $\sigma$, and all left-multiplications $L_y$ square to $\sigma^{-1}$. In the special case $\sigma = \mathrm{id}$ left-multiplications $L_y$ are simply involutions of $M \setminus \{y\}$. Thus, $N_{72}(n) \geq$ A000085$(n-1)^n = e^{(1/2)n^2 \log n + O(n^2)}$, see (31) for precise asymptotics. To find an upper bound, consider that for any choice of permutation $\sigma$ (of which there are $e^{O(n \log n)}$) one can choose each $L_y$ one entry at a time; each time we choose one entry another entry is automatically fixed, so that we have at most $n(n-2)(n-4)\dots$ choices of $L_y$.
- Law 75 ($x = y \diamond (y \diamond (y \diamond x))$): each left multiplication is chosen independently to be a permutation of order 1 or 3, so one has $N_{75}(n) =$ A001470$(n)^n = e^{(2/3)n^2(\log n - 1) + n^{4/3} + O(n \log n)}$.
- Law 63 ($x = y \diamond (x \diamond (x \diamond y))$) and Law 73 ($x = y \diamond (y \diamond (x \diamond y))$) have the same finite spectrum, which is likely asymptotically full, and the numbers may be calculable. The law admits explicit idempotent magmas of sizes 5,7,8,11 and more. The idea is then to partition the complete graph $K_n$ into cliques of these sizes, and use the corresponding magma operation for $x \diamond y$ belonging to a given clique. It may be possible to construct an idempotent magma of size 29 in this way from 6 copies of $K_7$ and 28 copies of $K_5$.

## 4.5 Order 3: second shape $\_\_ = \_\_ \diamond ((\_\_ \diamond \_\_) \diamond \_\_)$

For several laws below (law 101, 102, 104) we only determine a lower bound $e^{n^2(\log n - \log\log n)+O(n^2)}$ and do not rule out that these equations could be "vacuum" laws in the sense of having an $e^{n^2 \log n + O(n \log n)}$ asymptotics. However, the absence of square in these equations makes them appear more similar to gas laws than vacuum ones, so it is natural to expect the $e^{n^2(\log n - \log\log n)+O(n^2)}$ to hold.

- Law 99 ($x = x \diamond ((x \diamond x) \diamond x)$) is a single-variable law hence $N_{99}(n) \geq N_3(n) = n^{n^2-n}$.

- Law 109 ($x = x \diamond ((y \diamond y) \diamond z)$) and its specializations law 100 (where $y \to x$), law 107 (where $z \to x$) and law 108 (where $z \to y$) hold in magmas where $x \diamond x = 0$, $0 \diamond x = 0$, $x \diamond 0 = x$, and other products are unconstrained, so $N_{100}(n), N_{107}(n), N_{108}(n) \geq N_{109}(n) \geq n^{n^2-3n+3}$, where one factor of $n$ comes from the possibility to change 0 to any other element.

- Law 101–106, of the form $x = x \diamond ((\alpha \diamond \beta) \diamond \gamma)$ with $(\alpha, \beta, \gamma)$ being $(x, y, x)$, $(x, y, y)$, $(x, y, z)$, $(y, x, x)$, $(y, x, y)$, $(y, x, z)$, respectively, all hold in logarithmic projection magmas (16) hence $N_A(n) \geq e^{n^2(\log n - \log\log n) + O(n^2)}$ for these laws. In addition, law 103 implies law 9 and law 106 implies law 10, which puts upper bounds such that $N_{103}(n)$ and $N_{106}(n)$ both take the form $e^{n^2(\log n - \log\log n) + O(n^2)}$. Upper bounds for the other laws 101, 102, 104, 105 are **not available**.

- Law 111 ($x = x \diamond ((y \diamond z) \diamond y)$): let $m = \lfloor n / \log n \rfloor$ and set

$$\begin{aligned} &x \diamond 0 = x, \\ &x \diamond x = 0, \\ &x \diamond y = 0 \quad \text{for } x \in [\![0, m-1]\!] \text{ and } y \neq 0, \\ &x \diamond y \in [\![0, m-1]\!] \quad \text{for } x \in [\![m, n-1]\!] \text{ and } y \in [\![0, n-1]\!] \setminus \{x\}. \end{aligned} \tag{36}$$

  In particular, all $0 \diamond x = 0$. We check that all $(x \diamond y) \diamond x = 0$. This is immediate for $y = 0$, or ($x < m$ and $y \neq 0$), or $x = y$. For the remaining case $x \geq m$ and $y \neq x$ we have $x \diamond y < m$ and $x \neq 0$ so $(x \diamond y) \diamond x = 0$ too. Law 111 then immediately follows. The number of such magmas is **at least** $m^{(n-m)(n-1)} = e^{n^2(\log n - \log\log n) + O(n^2)}$.

- Law 115 ($x = y \diamond ((x \diamond x) \diamond y)$):

- Law 117 ($x = y \diamond ((x \diamond y) \diamond x)$): a technique initially developped for law 167 provides an **upper bound**.[3] Consider the map $\varphi \coloneqq (x, y) \mapsto (x \diamond y, x)$ and iterate it:

$$\varphi^{\circ 4}(x, y) = \varphi^{\circ 3}(x \diamond y, x) = \varphi^{\circ 2}\big((x \diamond y) \diamond x, x \diamond y\big) = \varphi\big(y, (x \diamond y) \diamond x\big) = (x, y), \tag{37}$$

  where we used law 2100 and law 117. The bijection $\varphi$ has cycles of lengths $1, 2, 4$. The magma can be redundantly described by: (1) stating for each $(x, y)$ whether it belongs to a shortened cycle, namely whether $\varphi^{\circ 2}(x, y) = (x, y)$, namely $x \diamond y = y$ and $y \diamond x = x$, then (2) specifying the entries of the

[3] In the class of right quasigroups (meaning that right multiplications are bijective), the right division operation defined by $(x/y) \diamond y = x$ obeys law 167 ($(x/y)/(y/x) = y$) if and only if $\diamond$ obeys law 117. This can most readily be seen by noting that law 117 is equivalent to law 2100 $x = ((y \diamond x) \diamond y) \diamond (y \diamond x)$, then setting $y = z/x$ so as to rewrite this law as $x = (z \diamond (z/x)) \diamond z$, which can be divided successively by $z$ and by $z/x$. We will not use this parastrophic equivalence between laws 117 and 167.

operation table successively for long cycles, with the values of $x \diamond y$ and of $(x \diamond y) \diamond x$ fixing two more products $((x \diamond y) \diamond x) \diamond (x \diamond y)$ and $y \diamond ((x \diamond y) \diamond x)$. Thus,

$$N_{117}(n) \le 2^{n^2} n^{2n^2/4} = e^{(1/2)n^2 \log n + O(n^2)}. \tag{38}$$

- Law 118 ($x = y \diamond ((x \diamond y) \diamond y)$): equivalent for finite magmas to law 73, parastrophically equivalent to the Dupont law 63, so $N_{118}(n) = N_{63}(n)$.

- Law 124 ($x = y \diamond ((y \diamond x) \diamond x)$): we construct magmas very similar to those for law 65, the only difference being that we want $(y \diamond x) \diamond x = y \diamond x$ instead of $x \diamond (y \diamond x) = y \diamond x$. We fix a surjection $\pi \colon M \to Q$ to some set. For each $x \in M$ and $i \in Q$ with $\pi(x) \ne i$, select an involution $f_{x,i} \colon \pi^{-1}(i) \to \pi^{-1}(i)$, and for $\pi(x) = i$ set $f_{x,i}(z) = x$ to be a constant map (here the magmas differ from those for law 65). Then define $x \diamond y = f_{x,\pi(y)}(y)$. By construction $\pi(x \diamond y) = \pi(y)$ so $x$ and $y \diamond x$ have the same image under $\pi$ and hence $(y \diamond x) \diamond x = y \diamond x$. Thus, $y \diamond ((y \diamond x) \diamond x) = y \diamond (y \diamond x) = f^{\circ 2}_{y,\pi(x)}(x) = x$. The number of magmas built this way, for a fixed $q$ and surjection whose preimages each have at least $\lfloor n/q \rfloor$ elements, is a power of the number $N_{13}$ of involutions, giving the **lower bound**

$$\begin{aligned} N_{124}(n) &\ge \big(N_{13}(\lfloor n/q \rfloor)\big)^{n(q-1)} \\ &\overset{q=\log n}{=} e^{(1/2)n^2(\log n - \log\log n) + O(n^2)}. \end{aligned} \tag{39}$$

- Law 125 ($x = y \diamond ((y \diamond x) \diamond y)$): equivalent for finite magmas to law 73, parastrophically equivalent to the Dupont law 63, so $N_{118}(n) = N_{63}(n)$.

- Law 127 ($x = y \diamond ((y \diamond y) \diamond x)$): such a magma is characterized by the involutive squaring map $S \colon x \mapsto x \diamond x$, for each fixed point of $S$ an involution (left-multiplication), and a bijection for each two-orbit, so

$$N_{127}(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!2^k k!} (N_{13}(n))^{n-2k} n!^k = e^{(1/2)n^2(\log n - 1) + O(n^{3/2})}. \tag{40}$$

To find the asymptotics we used that the number of involutions is $N_{13}(n) = n!^{1/2} e^{O(\sqrt{n})}$ so $(N_{13}(n))^{n-2k} n!^k = n!^{n/2} e^{O(n^{3/2})}$, which can be pulled out of the sum. The sum then counts the number of squaring maps, namely the number of involutions, $n!^{1/2} e^{O(\sqrt{n})}$.

- Law 138 ($x = y \diamond ((z \diamond y) \diamond x)$): in finite magmas, this law implies that left multiplications $L_y$ are bijective. Consider the equivalence relation $x \sim y$ if $L_x = L_y$ namely $\forall z, x \diamond z = y \diamond z$. To any equivalence class $C \subset M$ (and corresponding left multiplication $L_C$ defined as $L_x$ for any $x \in C$), we associate the "inverse" equivalence class $C' = \{y \mid L_y = L_C^{-1}\}$ whose left multiplication is the inverse of $L_C$. This inverse class is non-empty since the law states precisely that $z \diamond y \in C'$ for all $y \in C$ and $z \in M$. In addition, $C'' = C$. Thus, the magma is described by a choice of equivalence relation on $M$, of involution $C \mapsto C'$ (which may have fixed points), and of left multiplication maps $L_C$ that map each equivalence class $D$ to $D'$. Thus,

$$\begin{aligned} N_{138}(n) &= \sum_{k=1}^{n} \sum_{[\![0,n-1]\!]=I_1\sqcup\cdots\sqcup I_k} \sum_{\sigma\in S_k \text{ involution}} \prod_{i=1}^{k} |I_{\sigma(i)}|^{k|I_i|} \\ &\leq \mathrm{Bell}(n) N_{13}(n) \max_{1\leq k\leq n} \max_{n=n_1+\cdots+n_k} \prod_{i=1}^{k} n_i^{kn_i} \\ &\leq \mathrm{Bell}(n) N_{13}(n) \max_{1\leq k\leq n} (n-k+1)^{k(n-k+1)} = e^{(1/4)n^2\log(n/2)+o(n^2)}, \end{aligned} \tag{41}$$

with upper bounds found as follows. First, by monotonicity of log, for any $a, b > 0$ we have $(a-b)(\log a - \log b) > 0$ so $a^b b^a \leq a^a b^b$, so for a fixed partition the product is maximized when $\sigma$ is the identity. Then, by convexity of $n_i \mapsto n_i \log n_i$ the product is maximized for fixed $k$ by the most inhomogeneous partition $n = (n-k+1) + 1 + \cdots + 1$. The maximum over $k$ is then reached close to $k = n/2$. This discussion also gives a concrete construction of magmas saturating that asymptotic count: split the set $[\![0, n-1]\!] = [\![0, m-1]\!] \cup [\![m, n-1]\!]$ with $m = \lfloor n/2 \rfloor$, then set $x \diamond y = y$ for $y \in [\![0, m-1]\!]$ and set $x \diamond y = f_{\min(x,m)}(y)$ for $m+1$ maps $f_i \colon [\![m, n-1]\!] \to [\![m, n-1]\!]$. This gives $(n-m)^{(n-m)(m+1)}$ magma operations so $N_{138}(n) = e^{(1/4)n^2\log(n/2)+o(n^2)}$.

## 4.6 Order 3: third shape $\_ = (\_ \diamond \_) \diamond (\_ \diamond \_)$

- Law 151 ($x = (x \diamond x) \diamond (x \diamond x)$) states that squaring is an involution (counted by $N_{13}(n)$) and then off-diagonal entries are arbitrary, so $N_{151}(n) = n^{n^2-n} N_{13}(n) = e^{(n^2-n/2)\log n - n/2 + \sqrt{n} + O(1)}$.
- Law 152 ($x = (x \diamond x) \diamond (x \diamond y)$) and law 153 ($x = (x \diamond x) \diamond (y \diamond x)$) are obeyed by logarithmic projection magmas (16) hence $N_{152}(n), N_{153}(n) \geq e^{n^2(\log n - \log\log n)+O(n^2)}$.
- Law 159 ($x = (x \diamond y) \diamond (y \diamond x)$): the map $f \colon (x, y) \mapsto (x \diamond y, y \diamond x)$ is an involution of $M^2$ that commutes with $(x, y) \mapsto (y, x)$. Splitting into the diagonal $x = y$ and off-diagonal parts, we have an arbitrary squaring involution, and an involution of $n(n-1)$ elements that commutes with a given fixed-point-free involution, counted by A000898$(q) = e^{(1/2)q(\log(2q)-1)+\sqrt{2q}+O(1)}$,

$$N_{159}(n) = N_{13}(n)\text{A000898}(n(n-1)/2) = e^{(1/2)n^2(\log n - 1/2)+(1/2)n+\sqrt{n}+O(1)}. \tag{42}$$

- Law 162 ($x = (x \diamond y) \diamond (z \diamond x)$):
- Law 167 ($x = (y \diamond x) \diamond (x \diamond y)$) only has magmas of size $n \equiv 0, 1 \mod 4$. The squaring map is an involution. The map $f \colon (x, y) \mapsto (x \diamond y, y \diamond x)$, restricted to $x \neq y$, squares to the fixed-point-free involution $(x, y) \mapsto (y, x)$, so $f$ has

orbits of size 4. It is more convenient to consider the unordered pairs $\binom{M}{2}$ of distinct elements: the map $g\colon \{x,y\} \mapsto \{x \diamond y, y \diamond x\}$ is an involution without fixed point, and there are $(\binom{n}{2} - 1)!!$ such maps. The function $f$ is then reconstructed by having an additional binary choice of whether $f(x,y) = (z,w)$ or $(w,z)$ for each orbit $\{\{x,y\},\{w,z\}\}$ of $g$, Thus,

$$\begin{aligned} N_{167}(n) &= \Big(\frac{n(n-1)}{2} - 1\Big)!!\, 2^{\frac{n(n-1)}{4}} N_{13}(n) \\ &= e^{(1/2)n^2 \log n - (1/4)n(n+2) + \sqrt{n} + O(1)}. \end{aligned} \tag{43}$$

- Law 168 ($x = (y \diamond x) \diamond (x \diamond z)$) only has magmas of size $n = m^2$. We show in subsubsection 4.2.4 that

$$(1/8)n^{3/2} \log n + O(n^{3/2}) \le \log N_{168}(n)) \le (1/2)n^{3/2} \log n + O(n \log n). \tag{44}$$

## 4.7 Order 3: fourth shape $\_ \diamond \_ = \_ \diamond (\_ \diamond \_)$

All laws of the form $x \diamond x = \_ \diamond (\_ \diamond \_)$ are consequences of law 321 ($x \diamond x = y \diamond (z \diamond w)$), equivalent to law 314 ($x \diamond x = y \diamond (x \diamond z)$), whose precise counting and asymptotics we derive in (46). This puts a lower bound $N_A(n) \ge e^{n^2(\log n - \log\log n) + O(n^2)}$ for $A \in \{307, 308, \dots, 319\}$. These laws are thus at least pressureless gas laws. For some of them we find $e^{n^2 \log n + O(n \log n)}$ magmas, thus showing that they are in fact vacuum laws. For some we find an upper bound on $N_A(n)$, or even an explicit formula.

- Law 307 ($x \diamond x = x \diamond (x \diamond x)$) is a single-variable law so $N_{307}(n) = e^{n^2 \log n + O(n \log n)}$, but we can give an exact formula. Each $L_x$ is independent; either $L_x(x) = x$ and the other $n-1$ entries are arbitrary, or $L_x(x) \ne x$ and the value of $L_x$ at $L_x(x)$ is fixed but the other $n-2$ entries are arbitrary, so $N_{307}(n) = ((2n-1)n^{n-2})^n$.
- Law 308 ($x \diamond x = x \diamond (x \diamond y)$) has $N_{308}(n) = e^{n^2(\log n - \log\log n) + O(n^2)}$. The lower bound holds because 308 is implied by 314 as explained above, and alternatively because logarithmic projection magmas (16) obey law 308. We can actually give a precise count. Each $L_x$ is independent and described by the choice of $L_x(x) \in M$, and of subset $L_x^{-1}(L_x(x))$ containing $x$ and $L_x(x)$, and finally a choice of $L_x(y) \in L_x^{-1}(L_x(x)) \setminus \{L_x(x)\}$ for each $y \notin L_x^{-1}(L_x(x))$. If $L_x(x) = x$ then this is the same data as for each left multiplication for law 9, so we have A000248$(n-1)$ possibilities, see (28). If $L_x(x) \ne x$ then the number is slightly different: overall we find

$$\begin{aligned} N_{308}(n) &= \left(\sum_{k=1}^{n-1} \binom{n-1}{k} k^{n-1-k} + (n-1)\sum_{k=0}^{n-2} \binom{n-2}{k} (k+1)^{n-2-k}\right)^n \\ &= \left(\sum_{k=1}^{n-1} \binom{n-1}{k} k^{n-1-k}(k+1)\right)^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \end{aligned} \tag{45}$$

The asymptotics are found by noting that the sum, binomial coefficient, and $k^{-1}(k+1)$ factors can only contribute $e^{O(n^2)}$ factors, leaving us with $\max_{0\le k\le n} k^{n(n-k)}$, easily estimated.

- Law 310 ($x \diamond x = x \diamond (y \diamond y)$) and Law 312 ($x \diamond x = y \diamond (x \diamond x)$) and Law 316 ($x \diamond x = y \diamond (y \diamond y)$) are obeyed by magmas with $x \diamond x = 0$ and $x \diamond 0 = 0$, so $N_{310}(n), N_{312}(n), N_{316}(n) \ge n^{n^2-2n+1}$. We could give exact counts by noting that the magma operation is characterized by the squaring map $S\colon x \mapsto x \diamond x$ (idempotent for law 312, constant for law 316) and by the operation restricted to $x \diamond y$ for $y$ non-square and $x \neq y$.

- Law 311 ($x \diamond x = x \diamond (y \diamond z)$) has $N_{311}(n) \ge e^{n^2(\log n - \log\log n)+O(n^2)}$. The law is a consequence of law 314, which implies the lower bound. To get an upper bound, we note that the magma is determined by the subset $P = \{y \diamond z \mid y, z \in M\}$, by the squaring map with values in $P$, and by the operation restricted to $M \times (M \setminus P) \to P$ (which redundantly includes squares of $M \setminus P$), so

$$N_{311}(n) \le \sum_{k=1}^{n} \binom{n}{k} k^{k+n(n-k)} = e^{n^2(\log n - \log\log n)+O(n^2)}. \qquad (46)$$

- Law 314 ($x \diamond x = y \diamond (x \diamond z)$): this law is equivalent to law 321 stating that all squares and all products $\_ \diamond (\_ \diamond \_)$ are equal. The magma operation is characterized by the set of products $P = \{x \diamond y \mid x, y \in M\}$, by the joint value of all $x \diamond y$ for $y \in P$ or $x = y$, and by the operation restricted to $y \in M \setminus P$ and $x \neq y$, so

$$N_{314}(n) = \sum_{k=1}^{n} \binom{n}{k} k^{1+(n-k)(n-1)} = e^{n^2(\log n - \log\log n)+O(n^2)}. \qquad (47)$$

- Law 317 ($x \diamond x = y \diamond (y \diamond z)$): this law states that squares are all equal to some element $e$ and that each left multiplication $L_y$ squares to that constant. The data specifying $L_y$ is equivalent as that for law 9: instead of having $L_y \circ L_y$ taking the constant value $y$ and $L_y(y) = y$, we have $L_y \circ L_y$ and $L_y(y)$ taking the value $e$. Thus, we can reuse the exactly known $N_9(n)$ (28),

$$N_{317}(n) = nN_9(n) = e^{n^2(\log n - \log\log n)+O(n^2)}. \qquad (48)$$

- For Law 309 ($x \diamond x = x \diamond (y \diamond x)$), Law 313 ($x \diamond x = y \diamond (x \diamond y)$), Law 315 ($x \diamond x = y \diamond (y \diamond x)$), Law 318 ($x \diamond x = y \diamond (z \diamond x)$), Law 319 ($x \diamond x = y \diamond (z \diamond y)$), we content ourselves with a **lower bound** $N_A(n) \ge N_{314}(n) = e^{n^2(\log n - \log\log n)+O(n^2)}$. For law 309 this can alternatively be obtained from the logarithmic projection magmas (16). For law 315 it seems possible to give a closed formula by specifying first the squaring map $S\colon x \mapsto x \diamond x$, which is idempotent ($S \circ S = S$), and then each left multiplication $L_y$ separately, which must obey $L_y \circ L_y = S$.

- Law 323 ($x \diamond y = x \diamond (x \diamond y)$): this law states that each left multiplication (independently) is an idempotent map. Thus, in terms of the same OEIS sequence as in (28) (but shifted by 1),

$$N_{323}(n) = \big(\text{A000248}(n)\big)^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{49}$$

- Law 325 ($x \diamond y = x \diamond (y \diamond x)$):

- Law 326 ($x \diamond y = x \diamond (y \diamond y)$) is a consequence of the idempotence law 3 ($x = x \diamond x$), so $N_{326}(n) \geq N_3(n) = n^{n^2 - n}$.

- Law 327 ($x \diamond y = x \diamond (y \diamond z)$): consider the equivalence relation $x \sim y$ iff the right mutiplication maps coincide, $R_x = R_y$ (namely $\forall z, z \diamond x = z \diamond y$). We denote by $[x]$ the equivalence class and $R_{[x]}$ the common value of the right-multiplication maps. Law 327 states that $y \diamond z \sim y$, in other words $R_z$ maps each equivalence class $[y]$ to itself. To describe the magma, once the equivalence classes are fixed, one has to specify a collection of right multiplication maps $R_{[x]} \colon [y] \to [y]$ for each pair of equivalence classes. Thus (calculations are similar to those for law 138 in (41)):

$$\begin{aligned} N_{327}(n) &= \sum_{k=1}^{n} \sum_{[\![0,n-1]\!] = I_1 \sqcup \cdots \sqcup I_k} \prod_{i=1}^{k} |I_i|^{k|I_i|} \\ &\leq \operatorname{Bell}(n) \max_{1 \leq k \leq n} \max_{n = n_1 + \cdots + n_k} \prod_{i=1}^{k} n_i^{k n_i} \\ &\leq \operatorname{Bell}(n) \max_{1 \leq k \leq n} (n - k + 1)^{k(n-k+1)} = e^{(1/4) n^2 \log(n/2) + o(n^2)}, \end{aligned} \tag{50}$$

where we used that, for a given number $k$ of parts, the product is maximized by the most inhomogeneous partition $n = (n-k+1) + 1 + \cdots + 1$ because of convexity of $n_i \mapsto n_i \log n_i$. The maximum over $k$ is then reached close to $k = n/2$. This discussion also gives a concrete construction of magmas saturating that asymptotic count: split the set $[\![0, n-1]\!] = [\![0, m-1]\!] \cup [\![m, n-1]\!]$ with $m = \lfloor n/2 \rfloor$, then set $x \diamond y = x$ for $x \in [\![0, m-1]\!]$, and set $x \diamond y = f_{\min(y,m)}(x)$ for $m+1$ maps $f_i \colon [\![m, n-1]\!] \to [\![m, n-1]\!]$. This gives $(n-m)^{(n-m)(m+1)}$ magma operations so $N_{327}(n) = e^{(1/4)n^2 \log(n/2) + o(n^2)}$.

- Law 329 ($x \diamond y = x \diamond (z \diamond y)$): consider the equivalence relation $x \sim y$ iff $R_x = R_y$. Then $\operatorname{im}(R_x) \subset [x]$. The magma is specified by a choice of partition into equivalence classes, and then for each equivalence class $[x] \subset M$, a choice of right-multiplication mapping $M$ to $[x]$. This gives the upper bound (where the maximum is taken over all integer partitions of $n$)

$$\begin{aligned} N_{329}(n) &\leq \operatorname{Bell}(n) \max_{n = n_1 + \cdots + n_k} \prod_{i=1}^{k} n_i^n = \operatorname{Bell}(n) \big(\text{A000792}(n)\big)^n \\ &\leq e^{((\log 3)/3) n^2 + n \log n + O(n)} \end{aligned} \tag{51}$$

where we used observations from the OEIS that A000792$(3k) = 3^k$, A000792$(3k+1) = 3^{k-1} \times 4$, A000792$(3k+2) = 3^k \times 2$. This optimal value of the product also tells us how to reach essentially the same asymptotics by splitting $M$ into triplets specifically, so for $n = 3k + j$, $j \in \{0, 1, 2\}$ we find

$$N_{329}(n) \geq \frac{n!}{3!^k j!} 3^{kn} \max(j, 1)^n \geq e^{((\log 3)/3)n^2 + n \log n + O(n)}. \tag{52}$$

- Law 332 ($x \diamond y = y \diamond (x \diamond x)$) implies the commutative law 43 and is implied by idempotence together with commutativity, so $n^{n(n-1)/2} \leq N_{332}(n) \leq n^{n(n+1)/2}$.
- Law 333 ($x \diamond y = y \diamond (x \diamond y)$): the dual of this law is satisfied by logarithmic projection magmas (16) so we get the **lower bound** $N_{333}(n) \geq e^{n^2(\log n - \log\log n) + O(n^2)}$. Alternatively, law 343 implies this law, which gives the same asymptotic bound.
- Law 335 ($x \diamond y = y \diamond (y \diamond x)$):
- Law 336 ($x \diamond y = y \diamond (y \diamond y)$): this describes a set equipped with an idempotent self-map, so again the same OEIS sequence as for $N_9(n)$ in (28) shows up, but not raised to any power,

$$N_{336}(n) = \text{A000248}(n) = e^{n(\log n - \log\log n) + O(n)}. \tag{53}$$

  Modulo isomorphism, the magma is specified by a partition of $n$ (the sizes of preimages of each square), so $I_{336}$ is the partition number,

$$I_{336}(n) = \text{A000041}(n) = e^{\pi\sqrt{2n/3} + O(\log n)}. \tag{54}$$

- Law 343 ($x \diamond y = z \diamond (x \diamond y)$): the magma is characterized by the set $P$ of products and the operation on $M \times (M \setminus P)$ so

$$N_{343}(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n(n-k)} = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{55}$$

### 4.8 Selected higher-order laws

The following have not been verified as thoroughly.

- Law 412 ($x = x \diamond (x \diamond (x \diamond (x \diamond y)))$): one has $N_{412}(n) = \text{A000950}(n)^n$.
- Law 513 ($x = y \diamond (y \diamond (y \diamond (y \diamond x)))$): one has $N_{513}(n) = \text{A001472}(n)^n = e^{(3/4)n^2(\log n - 1) + n^{3/2}/2 + n^{5/4} + O(nlogn)}$.
- Law 3306 ($x \diamond y = x \diamond (x \diamond (x \diamond y))$): one has $N_{3306}(n) = \text{A060905}(n)^n$.

- Law 4268 ($x \diamond (x \diamond x) = x \diamond (x \diamond y)$): left-multiplications $L_x$ square to constant maps, almost like law 9 except that the constant also has to be chosen, so

$$N_{4268}(n) = n^n N_9(n) = (n\text{A000248}(n-1))^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{56}$$