THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY

A *projective plane* is a structure $\langle P, L, \in\rangle$ satisfying the axioms below. P is the set of *points*, while L is the set of *lines*. Lines are sets of points. The axioms:

(1) Every pair of distinct points lie on a unique line.
(2) Every pair of distinct lines intersect in a unique point.
(3) There exist 4 distinct points, no 3 on a line.

Example: Let D be a division ring, D^3 the left vector space over D, P the 1-dimensional subspaces, L its 2-dimensional subspaces.

The plane satisfies *Desargues' Law* if whenever a set of 6 points are perspective from a point, then they are perspective from a line. INSERT PICTURE IN YOUR MIND. The D^3 example is desarguean. There exist nondesarguean planes (but not higher-dimensional spaces).

Let a_0, a_1, a_2, b_0, b_1, b_2 be 6 points.

Perspective from a point: $(a_0 \vee b_0) \wedge (a_1 \vee b_1) \in a_2 \vee b_2$.

Let $c_k = (a_i \vee b_i) \wedge (a_j \vee b_j)$ for $\{i, j, k\} = \{0, 1, 2\}$. Perspective from a line is $c_2 \in c_0 \vee c_1$.

Now, given a plane Π, we try to reconstruct D following the way it appears in D^3. Choose 6 points a_1, a_2, a_3, c_{12}, c_{13}, c_{23} so that $(a_1 \vee a_3) \wedge (c_{12} \vee c_{13}) = c_{23}$, etc. Start with any a_1, a_2, c_{12}, c_{13} in general position and mimic how it works in vector spaces.

D will be the set of points on the line $a_1 \vee c_{12}$.

The *Fundamental Theorem of Projective Geometry* is that if Π is desarguean, then D with the defined operations is a division ring, and D coordinatizes Π.

The hard part is checking all the properties of a division ring.

1. How to prove the FTPG

. Let Π be a projective plane. Then Π is a complemented modular lattice. There are elements of height (or lattice dimension) 0 to 3:

- The empty set has height 0.
- Points have height 1, i.e., 1 above the bottom level.
- Lines have height 2.
- The whole plane has height 3.

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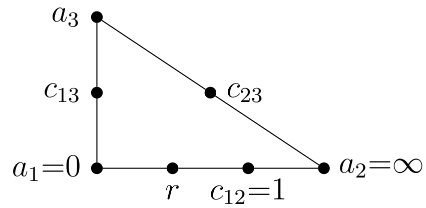

FIGURE 1. Setup

Use $x \vee y$ to denote the smallest flat containing x and y. For example, if x and y are points then $x \vee y$ is the line through them.

Use $p \wedge q$ to denote the largest flat contained in p and q, e.g., the point that is the intersection of two lines.

The modular law is: $x \geq y$ implies $x \wedge (y \vee z) = y \vee (x \wedge z)$. It is useful for calculations.

We coordinatize Π as follows. Let a_1, a_2, c_{13}, c_{23} be any 4 points, no two on a line. Define

$$a_3 = (a_1 \vee c_{13}) \wedge (a_2 \vee c_{23})$$
$$c_{12} = (a_1 \vee a_2) \wedge (c_{13} \vee c_{23})$$

Each of these new points is a meet of two lines from the original points.

Check that we have a *frame*:

$$a_i \vee a_j = a_i \vee c_{ij} = a_j \vee c_{ij}$$

Now we follow the prescription in the lattice theory notes, but pretending we know nothing.

- $D = \{x \in P : x \vee a_2 = a_1 \vee a_2 \text{ and } x \wedge a_2 = \varnothing\}$, that is, the division ring will consist of all points on the line $a_1 \vee a_2$ except a_2.
- $0 = a_1$ and $1 = c_{12}$.
- Define $r+s$, $-r$, and rs by the long formulas given in the lattice theory notes.
- You will also need r^{-1} for $r \neq a_1$ as the solution of some equation.
- Now check that D with these operations satisfies all the axioms of a division ring, e.g., $r + 0 = r$ etc.
- Cheat by looking in a projective geometry book. I recommend Fishback or M. K. Bennett, but there are probably lots.
- If you don't use Desargues' Law it won't work; you get a ternary ring. Marshall Hall's *Group Theory* is the classic source.

- Desargues' Law is a property that holds in some planes and not others.
- Do not try to prove commutativity of multiplication; that requires Pappus Theorem as an axiom.
- Do not attempt to borrow or steal my geometry books. Someone else has already beat you to it!

The operations are:

$$\begin{aligned} r+s &= [[(r \vee c_{13}) \wedge (a_2 \vee a_3)] \vee ([(s \vee a_3) \wedge (a_2 \vee c_{13})]] \wedge (a_1 \vee a_2) \\ -r &= [[[[(r \vee a_3) \wedge (a_1 \vee c_{23})] \vee a_2] \wedge (a_1 \vee a_3)] \vee c_{23}] \wedge (a_1 \vee a_2) \\ rs &= [[(r \vee c_{23}) \wedge (a_1 \vee a_3)] \vee [(s \vee c_{13}) \wedge (a_2 \vee a_3)]] \wedge (a_1 \vee a_2) \end{aligned}$$

Again, these are motivated by the vector space model:

$$\begin{aligned} A_1 &= D\langle 1,0,0\rangle \\ A_2 &= D\langle 0,1,0\rangle \\ A_3 &= D\langle 0,0,1\rangle \\ C_{12} &= D\langle 1,-1,0\rangle \\ C_{13} &= D\langle 1,0,-1\rangle \\ C_{23} &= D\langle 0,1,-1\rangle \\ \phi(r) &= D\langle 1,-r,0\rangle \end{aligned}$$

and the idea goes back to von Staudt (1847) and von Neumann.

2. Exercise: $r+s=s+r$

We assume the plane is Desarguean, and want to prove that the operations defined above give a division ring. The properties involving the constants, like $r+0=r$, are pretty easy.

For $r+s=s+r$ we first establish some notation:

$$\begin{aligned} L &= (r \vee c_{13}) \wedge (a_2 \vee a_3) \\ R &= (s \vee a_3) \wedge (a_2 \vee c_{13}) \\ L' &= (s \vee c_{13}) \wedge (a_2 \vee a_3) \\ R' &= (r \vee a_3) \wedge (a_2 \vee c_{13}) \end{aligned}$$

so that

$$\begin{aligned} r+s &= (L \vee R) \wedge (a_1 \vee a_2) \\ s+r &= (L' \vee R') \wedge (a_1 \vee a_2). \end{aligned}$$

This is illustrated in the top and bottom halves of Figure 2, respectively.

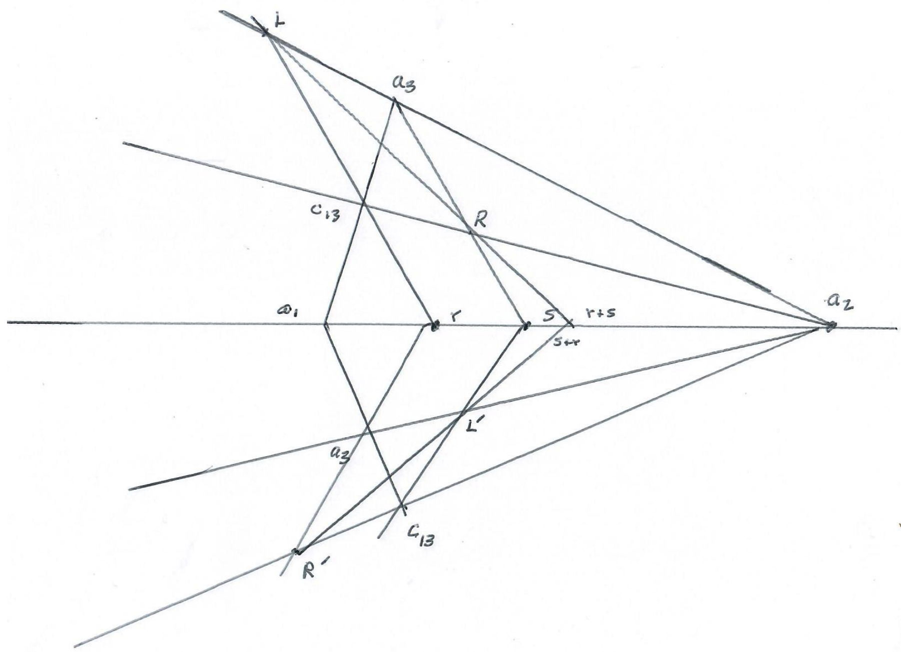

FIGURE 2. The top half shows $r + s$, the bottom half $s + r$. The points a_3 and c_{13} are duplicated.

Now we look at the triangles in Figure 3. In the top half we have the triangles $\langle R', a_3, R\rangle$ and $\langle L, c_{13}, L'\rangle$. These are axially perspective:

$$(R' \vee a_3) \wedge (L \vee c_{13}) = r$$
$$(R \vee a_3) \wedge (L' \vee c_{13}) = s$$
$$(R' \vee R) \wedge (L \vee L') = a_2$$

which are colinear. Hence they are centrally perspecitive: the lines R‘ $\vee L$, $a_3 \vee c_{13}$, and $R \vee L'$ meet in a point P.

Now consider the triangles $\langle R, c_{13}, L\rangle$ and $\langle L', a_2, R'\rangle$ in the bottom half of the figure. By the first triangle, they are centrally perspective through the point P, and hence axially perspective by Desargues'

Theorem. Now

$$(R \vee c_{13}) \wedge (L' \vee a_3) = a_2$$
$$(L \vee c_{13}) \wedge (R' \vee a_3) = r$$

so the point $(R \vee L) \wedge (R' \vee L')$ is on the line $a_2 \vee r = a_1 \vee a_2$, as desired.

This is left as an exercise in Fishback. It is not hard to check the proof we have just written, but it is not at all obvious what pairs of triangles to consider, and that is where perhaps a prover could do better than a human.

And most of the remaining axioms for a division ring, such as associativity and distributivity, are harder, not easier!

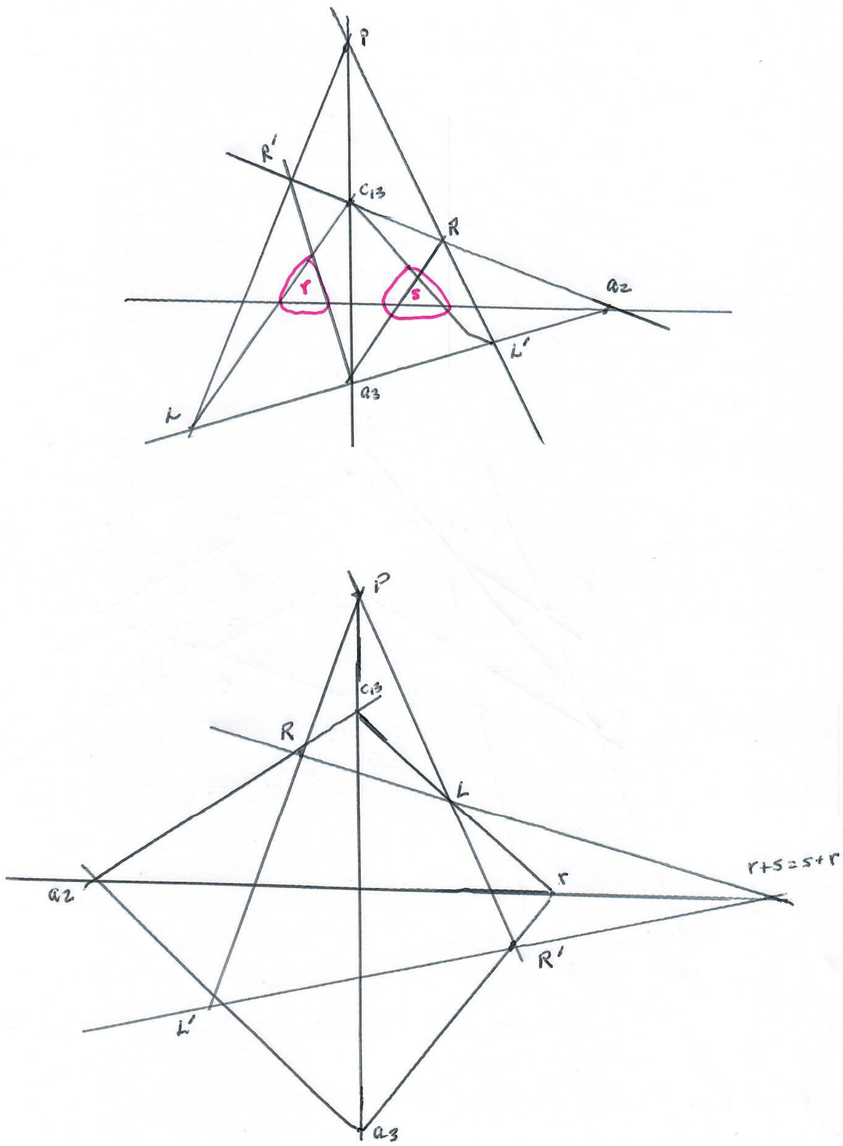

FIGURE 3. These are the two Desarguean triangles used in the proof that $r + s = s + r$.