HUMANIZED PROVER9 PROOF OF 650 $\implies xy = x$

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1. Introduction

Calling this proof "humanized" is a bit of a stretch. Since it is merely a proof that a magma satisfying one identity must satisfy another identity, there is no background theory to use and there are certainly no larger ideas at play. In this case, "humanization" just means making the steps of an equational proof explicit enough for a suitably patient human to follow. I do not claim that I have carried this out optimally.

The first proof Prover9 found was considerably longer and included some quite large identities, some of which had 6 variables. The shorter proof found below – with identities of reasonable length and no more than 5 variables – was found using the methodology described in [1]. In some cases, I reduced the number of variables in an identity by hand because it didn't need so many when it was used later.

The Prover9 proof itself has 27 steps. The proof below has 18 "steps" because I was able to consolidate a few of them.

2. The humanized proof

Starting with

$$x[y((zx)y)] = x\,, \tag{T650}$$

our goal is to prove

$$xy = x\,. \tag{LZ}$$

Step 1. In (T650), set $z = u$ and $x = v((wu)v)$, and then rewrite the left side using (T650) itself:

$$(v((wu)v))[y(\underbrace{u[v((wu)v)]}_{}y)] \overset{(T650)}{=} (v((wu)v))[y(\overbrace{\ u\ }^{}y)]$$

Thus by (T650) and a change of variables,

$$[x((yz)x)](u(zu)) = x((yz)x). \tag{1}$$

Step 2. In (T650), replace y with $y((w(zx))y)$ and rewrite the left side using (T650) itself:

$$x\{[y((w(zx))y)][\underbrace{(zx)[y((w(zx))y)]}_{}]\} \overset{(T650)}{=} x\{[y((w(zx))y)][\overbrace{zx}^{}]\}$$

Thus by (T650) and a change of variables,

$$x\{[y((z(ux))y)](ux)\} = x. \tag{2}$$

Step 3. In (2), replace x with $(xz)u$ and rewrite the left side using (T650):

$$[(xz)u]\{[y((\underbrace{z(u[(xz)u])}_{})y)](u[(xz)u])\} \overset{(T650)}{=} [(xz)u]\{[y(\overbrace{\ z\ }^{}y)](u[(xz)u])\}$$

Thus by (2) and a change of variables,

$$((xy)z)\{(u(yu))[z((xy)z)]\} = (xy)z. \tag{3}$$

Step 4. In (T650), set $x = t(st)$ and $z = q((rs)q)$, and then rewrite the left side using (1):

$$[t(st)](y((\underbrace{[q((rs)q)][t(st)]})y)) \overset{(1)}{=} [t(st)](y((\overbrace{q((rs)q)})y)).$$

Thus by (T650) and a change of variables,

$$(x(yx))(z((u((wy)u))z)) = x(yx). \tag{4}$$

Step 5. In (2), set $x = tu$, $z = r(s(tr))$, and then rewrite the left side using (1):

$$(tu)\{[y((\underbrace{[r(s(tr))](u(tu))})y)](u(tu))\} \overset{(1)}{=} (tu)\{[y(\overbrace{[r(s(tr))]}y)](u(tu))\}$$

Thus by (2) and a change of variables,

$$(xy)[\{z([u((wx)u)]z)\}(y(xy))] = xy. \tag{5}$$

Step 6. In (2), set $z = x$, $u = rs$, $y = t(st)$, and then rewrite the left side using (1):

$$x\{[(t(st))\{\underbrace{(x((rs)x))(t(st))}\}]((rs)x)\} \overset{(1)}{=} x\{[(t(st))\{\overbrace{x((rs)x)}\}]((rs)x)\}$$

Thus by (2) and a change of variables,

$$x(((y(zy))(x((uz)x)))((uz)x)) = x. \tag{6}$$

Step 7. In (4), set $z = s(ys)$ and rewrite the left side using (1):

$$(x(yx))([s(ys)]\{\underbrace{(u((wy)u))[s(ys)]}\}) \overset{(1)}{=} (x(yx))([s(ys)]\{\overbrace{u((wy)u)}\})$$

Thus by (4) and a change of variables,

$$(x(yx))((z(yz))(u((wy)u))) = x(yx). \tag{7}$$

Step 8. In (T650), set $z = (tu)v$, $x = (w(uw))(v((tu)v))$, and rewrite the left side using (3):

$$[(w(uw))(v((tu)v))][y((\underbrace{[(tu)v][(w(uw))(v((tu)v))]})y)] \overset{(3)}{=} [(w(uw))(v((tu)v))][y((\overbrace{(tu)v})y)]$$

Thus by (T650) and a change of variables,

$$((x(yx))(z((uy)z)))(w(((uy)z)w)) = (x(yx))(z((uy)z)). \tag{8}$$

Step 9. In (5), replace z with $z(xz)$, then set $z = w = y$, and then rewrite the left side using (1):

$$(xy)[\{(y(xy))[\underbrace{(u((yx)u))(y(xy))}]\}(y(xy))] \overset{(1)}{=} (xy)[\{(y(xy))[\overbrace{u((yx)u)}]\}(y(xy))]$$

Thus by (5),

$$(xy)[\{(y(xy))[u((yx)u)]\}(y(xy))] = xy. \tag{9}$$

Step 10. In (8), set $z = y = x$ and $w = x((xx)x)$, and rewrite the left side using (1):

$$\begin{aligned} &[(x(xx))(x((xx)x))][(x((xx)x))(((xx)x)(x((xx)x)))] \\ &= [(x(xx))(x((xx)x))][\underbrace{(x((xx)x))(t(xt))}] \\ &\qquad \text{where } t = (xx)x \\ &\overset{(1)}{=} [(x(xx))(x((xx)x))][\overbrace{x((xx)x)}]. \end{aligned}$$

Thus by (8),

$$[(x(xx))(x((xx)x))][x((xx)x)] = (x(xx))(x((xx)x)). \tag{10}$$

Step 11. In (T650), set $x = y = u((uu)u)$ and $z = u(uu)$, and then rewrite the left side using (10):

$$\begin{aligned} &(u((uu)u))[(u((uu)u))\{\underbrace{[(u(uu))(u((uu)u))](u((uu)u))}\}] \\ \stackrel{(10)}{=}\ &(u((uu)u))[(u((uu)u))\{\overbrace{(u(uu))(u((uu)u))}\}]. \end{aligned}$$

Thus by (T650) and a change of variables,

$$(x((xx)x))[(x((xx)x))\{(x(xx))(x((xx)x))\}] = x((xx)x). \tag{11}$$

Step 12. In (7), set $w = z = x$, $y = xx$ and $u = x((xx)x)$, and then rewrite the left side using (11):

$$(x((xx)x))[\underbrace{(x((xx)x))\{(x((xx)x))[(x(xx))(x((xx)x))]}]\}] \stackrel{(11)}{=} (x((xx)x))[\overbrace{x((xx)x)}].$$

Thus by (7),

$$(x((xx)x))(x((xx)x)) = x((xx)x). \tag{12}$$

Step 13. In (9), replace x with xx, set $y = x$ and $u = x((xx)x)$, and then rewrite the left side using (11) and (12):

$$\begin{aligned} &((xx)x)[\{\underbrace{(x((xx)x))[(x((xx)x))((x(xx))(x((xx)x)))]}\}(x((xx)x))] \\ \stackrel{(11)}{=}\ &((xx)x)[\underbrace{\{\overbrace{(x((xx)x))}\}(x((xx)x))}] \\ \stackrel{(12)}{=}\ &((xx)x)(\overbrace{x((xx)x)}). \end{aligned}$$

Thus by (9),

$$((xx)x)(x((xx)x)) = (xx)x \tag{13}$$

Step 14. In (3), set $u = z = y = z$ and rewrite the left side using (13):

$$((xx)x)\,\underbrace{((x(xx))(x((xx)x)))} \stackrel{(13)}{=} ((xx)x)(\overbrace{(xx)x})$$

Thus by (3),

$$((xx)x)((xx)x) = (xx)x\,. \tag{14}$$

Step 15. In (6), set $y = (xx)x$, $u = z = x$, and rewrite the left side using (13) and (14):

$$\begin{aligned} &x((\underbrace{(((xx)x)(x((xx)x)))}(x((xx)x)))((xx)x)) \\ \stackrel{(13)}{=}\ &x((\underbrace{(\overbrace{(xx)x})(x((xx)x))})((xx)x)) \\ \stackrel{(13)}{=}\ &x(\underbrace{(\overbrace{(xx)x})((xx)x)}) \\ \stackrel{(14)}{=}\ &x(\overbrace{(xx)x}) \end{aligned}$$

Thus by (6),

$$x((xx)x) = x \tag{15}$$

Step 16. In (T650), set $y = z = x$ and rewrite the left side using (15):

$$x(\underbrace{x((xx)x)}) \overset{(15)}{=} x\overbrace{\;x\;}\,.$$

Thus by (T650),

$$xx = x. \tag{16}$$

Step 17. In (T650), set $y = zx$ and rewrite the left side using (16):

$$x((zx)(\underbrace{(zx)(zx)})) \overset{(16)}{=} x(\underbrace{(zx)(\overbrace{\;zx\;})}) \overset{(16)}{=} x(\overbrace{\;zx\;})\,.$$

Thus by (T650),

$$x(zx) = x. \tag{17}$$

Step 18. Finally, apply (17) to the left side of (T650):

$$x(\underbrace{y((zx)y)}) \overset{(17)}{=} x\overbrace{\;y\;}\,.$$

Thus by (T650), (LZ) follows.

3. The Prover9 proof

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1 x * y = x # label(non_clause) # label(goal).  [goal].
2 x * (y * ((z * x) * y)) = x # label("650").  [assumption].
3 c1 * c2 != c1.  [deny(1)].
4 (x * ((y * z) * x)) * (u * (z * u)) = x * ((y * z) * x).  [para(2(a,1),2(a,1,2,2,1))].
5 x * ((y * ((z * (u * x)) * y)) * (u * x)) = x.  [para(2(a,1),2(a,1,2,2))].
6 ((x * y) * z) * ((u * (y * u)) * (z * ((x * y) * z))) = (x * y) * z.
  [para(2(a,1),5(a,1,2,1,2,1))].
7 (x * (y * x)) * (z * ((u * ((w * y) * u)) * z)) = x * (y * x).
  [para(4(a,1),2(a,1,2,2,1))].
8 (x * y) * ((z * ((u * ((w * x) * u)) * z)) * (y * (x * y))) = x * y.
  [para(4(a,1),5(a,1,2,1,2,1))].
9 x * (((y * (z * y)) * (x * ((u * z) * x))) * ((u * z) * x)) = x.
  [para(4(a,1),5(a,1,2,1,2))].
10 (x * (y * x)) * ((z * (y * z)) * (u * ((w * y) * u))) = x * (y * x).
   [para(4(a,1),7(a,1,2,2))].
11 ((x * (y * x)) * (z * ((u * y) * z))) * (w * (((u * y) * z) * w))
   = (x * (y * x)) * (z * ((u * y) * z)). [para(6(a,1),2(a,1,2,2,1))].
12 (x * y) * (((z * (x * z)) * (u * ((w * x) * u))) * (y * (x * y))) = x * y.
   [para(4(a,1),8(a,1,2,1,2))].
13 ((x * (y * x)) * (y * ((z * y) * y))) * (y * ((z * y) * y))
   = (x * (y * x)) * (y * ((z * y) * y)). [para(4(a,1),11(a,1,2))].
14 (x * ((y * x) * x)) * ((x * ((y * x) * x)) * ((z * (x * z)) * (x * ((y * x) * x))))
   = x * ((y * x) * x). [para(13(a,1),2(a,1,2,2))].
15 (x * ((y * y) * x)) * (y * ((y * y) * y)) = x * ((y * y) * x).  [para(14(a,1),10(a,1,2))].
16 ((x * x) * y) * ((x * ((x * x) * x)) * (y * ((x * x) * y))) = (x * x) * y.
   [para(14(a,1),12(a,1,2,1))].
17 ((x * x) * x) * (x * ((x * x) * x)) = (x * x) * x.  [para(15(a,1),16(a,1,2))].
18 x * ((((y * y) * y) * (x * ((z * y) * x))) * ((z * y) * x)) = x.
   [para(17(a,1),9(a,1,2,1,1))].
19 ((x * y) * z) * (((y * y) * y) * (z * ((x * y) * z))) = (x * y) * z.
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   [para(17(a,1),6(a,1,2,1))].
20 x * (((x * x) * x) * ((x * x) * x)) = x.  [para(17(a,1),18(a,1,2,1))].
21 ((x * x) * x) * ((x * x) * x) = (x * x) * x.  [para(17(a,1),19(a,1,2))].
22 x * ((x * x) * x) = x.  [para(21(a,1),20(a,1,2))].
23 x * x = x.  [para(22(a,1),2(a,1,2))].
24 x * (x * x) = x.  [para(23(a,1),22(a,1,2,1))].
25 x * (y * x) = x.  [para(24(a,1),2(a,1,2))].
26 x * y = x.  [para(25(a,1),2(a,1,2))].
27 $F.  [resolve(26,a,3,a)].
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References

[1] M. Kinyon, Proof simplification and automated theorem proving, *Philos. Trans. Roy. Soc. A* **377** (2019), no. 2140, 20180034, 9 pp.