

A formalization of the change of variables formula for integrals in mathlib

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Abstract. We report on a formalization of the change of variables formula in integrals, in the mathlib library for Lean. Our version of this theorem is extremely general, and builds on developments in linear algebra, analysis, measure theory and descriptive set theory. The interplay between these domains is transparent thanks to the highly integrated development model of mathlib.

Keywords: Change of variables · Integral · Formalization · mathlib.

1 Introduction

The change of variables formula in integrals is a basic tool in mathematics, playing an important role both in concrete computations of integrals and in more theoretical domains, notably Poincaré duality for de Rham cohomology. Its most basic formulation is the following:

Theorem 1.1. *Consider the vector space \mathbb{R}^n with its standard Lebesgue measure, and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a C^1 -diffeomorphism (i.e., f is a bijection, it is continuously differentiable, and so is its inverse). Then, for any integrable function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, the function $x \mapsto |\det Df(x)| \cdot g(f(x))$ is also integrable and*

$$\int g(y) \, d\text{Leb}(y) = \int |\det Df(x)| \cdot g(f(x)) \, d\text{Leb}(x).$$

This paper is devoted to the description of a formalization of (a more sophisticated version of) this theorem, in the Lean proof assistant, developed at Microsoft Research by Leonardo de Moura [9], within the library `mathlib` [11]. Apart from its mathematical relevance, an interest of this theorem from the formalization point of view is that it mixes several domains of mathematics that are typically taught in different courses, notably linear algebra, calculus, measure theory (and descriptive set theory for the aforementioned more sophisticated version). Therefore, it can only be formalized in a library which is developed enough in all these directions, and in which all these areas can interact in a coherent way. This is the case of `mathlib`, but also of the main library of Isabelle/HOL (which already contains a version of the above theorem) or of `mathcomp-analysis` in Coq (which does not contain a version of the above theorem at the time of this

writing, but might in the near future). The need for such coherence in advanced mathematics libraries will be a guiding theme in this paper.

This paper is written both for mathematicians who want to learn more on advanced versions of the change of variables formula or on theorem provers, and for formalizers: we will explain design issues that show up at different places, and justify the specific choices that have been made in the `mathlib` formalization to solve these issues.

2 Sketch of proof of Theorem 1.1

Let us sketch a proof of Theorem 1.1 as may be found in standard textbooks, to highlight the tools that are needed. Approximating the function g by characteristic functions of measurable sets, and then the measurable set by a compact set, it is sufficient to prove the following statement: if K is a compact set, then

$$\text{Leb}(f(K)) = \int_K |\det Df(x)| \, d\text{Leb}(x).$$

We will check that each of these quantities is bounded above by the other one.

Fix $\delta > 0$. Let $\varepsilon > 0$. Cover K by boxes made from a grid of mesh ε , and denote the center of such a box B_i by c_i . By uniform continuity of Df , if ε is small enough, the differential Df is arbitrarily close to $Df(c_i)$ on B_i . It follows that $f(B_i)$ is included in the image of B_i under the linear map $(1 + \delta)Df(c_i)$, uniformly in i . Then

$$\begin{aligned} \text{Leb}(f(K)) &\leq \sum_i \text{Leb}(f(B_i)) \leq \sum_i \text{Leb}((1 + \delta)Df(c_i)(B_i)) \\ &= (1 + \delta)^n \sum_i |\det(Df(c_i))| \text{Leb}(B_i), \end{aligned}$$

where the last equality follows from the fact that a matrix A rescales the volume according to $|\det A|$. If δ is small enough, then $|\det(Df(c_i))| \leq (1 + \delta)|\det Df(x)|$ for any $x \in B_i$ by uniform continuity. Then the above sum can be bounded by

$$(1 + \delta)^n \sum_i \int_{B_i} (1 + \delta) |\det Df(x)| \, d\text{Leb}(x).$$

Finally, denoting by K_ε the neighborhood of K given by the union of the B_i , we have proved that

$$\text{Leb}(f(K)) \leq (1 + \delta)^{n+1} \int_{K_\varepsilon} |\det Df(x)| \, d\text{Leb}(x)$$

When ε tends to 0, then K_ε tends to K . The dominated convergence theorem shows that the integral over K_ε converges to the integral over K . Finally, as δ is arbitrary, we have proved the inequality

$$\text{Leb}(f(K)) \leq \int_K |\det Df(x)| \, d\text{Leb}(x).$$

The converse inequality can be proved along the same lines, but one step is more delicate: one should show that $\text{Leb}(f(B_i)) \geq (1 + \delta)^{-n} \text{Leb}(Df(c_i)(B_i))$, i.e., one should show that $f(B_i)$ is comparatively large. A new ingredient is needed there, to prove that f is locally surjective (in a quantitative way). This follows from the local inverse theorem. Equivalently, one can mimic the above computation but for the map f^{-1} (which is also a diffeomorphism). This concludes the proof. \square

3 A more sophisticated version of the theorem

From the proof sketch above, it is obvious that some assumptions in Theorem 1.1 may be relaxed. For instance, it is not necessary that f is defined on the whole space: if it is a diffeomorphism between two open sets of \mathbb{R}^n , then the same proof will go through. It is also not necessary that the vector space is \mathbb{R}^n : any finite-dimensional real vector space with a Lebesgue measure (i.e., a translation invariant sigma-finite measure) will do, as such a measure has the same rescaling properties under linear maps.

On the other hand, since the proof relies on the local inverse theorem, it looks as though assuming that the map is defined on an open set and its differential is continuous and invertible can not be avoided. It turns out that this is not the case (and this may come as a surprise even to mathematicians who are very familiar with this theorem and its applications): all these assumptions can be dispensed with.

The most general version of the theorem is expressed in terms of a (slightly non-standard) notion of differentiability along a set:

Definition 3.1. *Consider a normed real vector space E , a map $f : E \rightarrow E$ and a continuous linear map $A : E \rightarrow E$. We say that A is a derivative of f at a point $x \in E$ along a set $s \subseteq E$ if, when y tends to x inside s , then $f(y) = f(x) + A(y - x) + o(y - x)$.*

When s is the whole space (or a neighborhood of x), this coincides with the usual notion of differentiability at x .

We say that a measure on a finite-dimensional real vector space is a *Lebesgue measure* if it is nonzero, sigma-finite, and invariant under left-translation. Such a measure always exists, and it is unique up to scalar multiplication. Such measures are also known as Haar measures in the more general context of locally compact topological groups.

Here is the general version of the change of variable formula.

Theorem 3.2. *Let E be a finite-dimensional normed real vector space endowed with a Lebesgue measure, $f : E \rightarrow E$ a map, s a Borel-measurable subset of E and $f'(x)$ a linear map on E for each x . Assume that f is injective on s , and that at every $x \in s$ then $f'(x)$ is a derivative of f at x along s . Then $f(s)$ is also Borel-measurable, and any function $g : E \rightarrow \mathbb{R}$ satisfies the equality*

$$\int_{f(s)} g(y) \, d\text{Leb}(y) = \int_s |\det f'(x)| \cdot g(f(x)) \, d\text{Leb}(x).$$

As in the proof sketch of Paragraph 2, this follows from a result on the measure of the image set:

$$\text{Leb}(f(s)) = \int_s |\det f'(x)| \, d\text{Leb}(x). \quad (1)$$

This theorem is proved in [4, Theorem 263D], with the difference that the emphasis there is on Lebesgue-measurable sets more than Borel-measurable ones. To obtain the fact that $f(s)$ is Borel-measurable if s is, one needs additionally the Lusin-Souslin theorem [5, Theorem 423I], an important and nontrivial result in descriptive set theory.

We will not give a full proof of Theorem 3.2, and refer the interested reader to [4] instead. Let us only stress where one can follow the sketch given in Section 2, and where one should depart from it. Let us focus on the proof of (1). Fix a small $\varepsilon > 0$. With the definition of differentiability along s , one may split s into countably many disjoint small sets s_i on which f is well approximated by a linear map A_i , up to ε . Then one would like to say that $\text{Leb}(f(s_i))$ is close to $|\det(A_i)| \cdot \text{Leb}(s_i)$, but one can not resort to the local inverse theorem.

For the direct inequality

$$\text{Leb}(f(s_i)) \leq (|\det(A_i)| + \varepsilon) \cdot \text{Leb}(s_i), \quad (2)$$

we fix $\delta > 0$ and we use a covering lemma (such as the Vitali or the Besicovitch covering theorems) ensuring that one can cover s_i with countably many balls $(B_{ij})_{j \in \mathbb{N}}$ whose measures add up to at most $\text{Leb}(s_i) + \delta$. For each such ball B_{ij} , its image has measure bounded by $(|\det(A_i)| + \varepsilon) \text{Leb}(B_{ij})$. Adding these estimates and letting δ tend to 0, we get (2). More on covering theorems in Section 6.

For the converse inequality

$$(|\det(A_i)| - \varepsilon) \cdot \text{Leb}(s_i) \leq \text{Leb}(f(s_i)),$$

there is nothing to prove if A_i is not invertible. If it is invertible, one argues that f is invertible on s_i and that its inverse is close to A_i^{-1} (this is a nonstandard version of the local inverse theorem). Then, one repeats the above computation for f^{-1} to get the desired estimate.

Adding all these inequalities over i one gets that $\text{Leb}(f(s)) = \sum_i \text{Leb}(f(s_i))$ is comparable to $\sum |\det(A_i)| \text{Leb}(s_i)$, which is comparable to $\int_s |\det f'(x)| \, d\text{Leb}(x)$ as $f'(x)$ is close to A_i on s_i . This concludes the proof. \square

A difficulty that we have ignored in this proof sketch is that the derivative along a set is in general not unique at points where the set is not fat enough. This has the unpleasant consequence that, in the statement of Theorem 3.2, the function $|\det f'(x)|$ is in general not Borel-measurable along s , as illustrated by the following example.

Example 3.3. Take $s = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$, and $f(x, y) = (x, 0)$. Let also t be any (possibly non-measurable) subset of \mathbb{R} , and set

$$f'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1_t(x) \end{pmatrix}.$$

Along horizontal directions, f' acts as the identity, so it is indeed a derivative of f along s . But $\det f'(x, y) = 1_t(x)$ is not a measurable function.

Nevertheless, Theorem 3.2 is still true in this example, since s and $f(s)$ have zero measure, so both the left and the right hand side vanish. In general, $f'(x)$ is uniquely defined on a full measure subset of s (its *Lebesgue density points*, which are again studied using covering lemmas), and is measurable there. In particular, even though $|\det f'(x)|$ is not always Borel-measurable, it coincides almost everywhere with a Borel-measurable function.

For Theorem 3.2 to be true, it means that the theory of integration one uses should work smoothly with functions which are not Borel-measurable, but coincide almost everywhere with a Borel-measurable function. While this was not the case of the first definition of the integral in `mathlib`, it has already been refactored (for different reasons) before the start of this project to allow non-Borel measurable functions, so no modification of the library was needed on this side.

Let us explain why the definition of integral had been refactored prior to this work. `mathlib` initially contained a definition of the integral for which the integral of non-measurable functions was zero by definition. This seemed quite satisfactory and made it possible to prove many theorems, until the formalization of the fundamental theorem of calculus. This theorem reads as follows: if two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous on an interval $[a, b]$ and g is the derivative of f there, then $\int g(x) \, d\text{Leb}_{|[a,b]}(x) = f(b) - f(a)$. It turns out that this theorem as stated was not true in `mathlib`: we are not making any assumption on g outside $[a, b]$, so there is no reason why g should be measurable globally, which means that $\int g(x) \, d\text{Leb}_{|[a,b]}(x)$ could be equal to 0 for no good reason. The first version of this theorem in `mathlib` therefore needed an additional assumption that g was measurable globally.

Without this assumption, it is true that g is null-measurable (i.e. almost everywhere measurable) with respect to the restricted measure $d\text{Leb}_{|[a,b]}$: it coincides almost everywhere with a measurable function, namely the function equal to g on $[a, b]$ and to 0 elsewhere (which is indeed measurable by continuity of g on $[a, b]$).

To get a more satisfactory statement for the fundamental theorem of calculus, the definition of the integral in `mathlib` was therefore refactored to allow for almost everywhere measurable functions: if a function is almost everywhere measurable, then its integral is defined to be the integral of a measurable function which coincides with it almost everywhere, and otherwise the integral is defined to be 0.

This definition has several advantages. For instance, if two functions coincide almost everywhere then they have the same integral. Moreover, it is exactly the kind of integration theory which is needed for Theorem 3.2 to hold! That this change was needed would not have been noticed in a pure measure-theory library, and was really a consequence of the interaction of different domains of mathematics in `mathlib`.

Getting definitions right the first time is hard. Definitions should be driven by the theorems they enable, even in different domains, and one should not be afraid to refactor a core definition.

As an aside, let us note that the definition of integrals was refactored a third time in `mathlib`, to allow for functions that take values in spaces which are not second-countable. While the standard definition of integration (writing a function as a pointwise limit of simple functions) is easier to work out when the target space is second-countable, this restriction prevents some applications to complex analysis and spectral theory. When these limitations were noticed, the definition was changed again, for the better. Now, the functions that can be integrated in `mathlib` are the almost everywhere strongly measurable ones, i.e., the functions that coincide almost everywhere with a pointwise limit of simple functions. And there are several results ensuring that most concrete functions are almost everywhere strongly measurable functions – for instance measurable functions into second-countable Banach spaces, or continuous functions from second-countable spaces.

4 The formalized version of the theorem

Here is the full statement of the formalized version of Theorem 3.2.

```

theorem integral_image_eq_integral_abs_det_fderiv_smul
  [normed_group E] [normed_space ℝ E] [finite_dimensional ℝ E]
  [measurable_space E] [borel_space E]
  (μ : measure E) [is_add_haar_measure μ]
  [normed_group F] [normed_space ℝ F] [complete_space F]
  {s : set E} {f : E → E} {f' : E → (E →L[ℝ] E)} (hs : measurable_set s)
  (hf' : ∀ x ∈ s, has_fderiv_within_at f (f' x) s x)
  (hf : set.inj_on f s) (g : E → F) :
  ∫ y in f '' s, g y ∂μ = ∫ x in s, |(f' x).det| · g (f x) ∂μ

```

Here is a rephrasing of the theorem for readers who are not familiar with Lean's syntax. We start with a finite-dimensional real normed vector space E , a measure μ on E which is assumed to be a Lebesgue measure (in more formal terms, an additive Haar measure), a subset s of E which is assumed to be measurable (assumption `hs`), and a function $f : E \rightarrow E$ which is injective on s (assumption `hf`). Consider also, for each $x \in E$, a continuous linear map $f'(x)$ on E . Assume that, for each $x \in s$, then $f'(x)$ is a derivative of f at x along s (assumption `hf'`). Then the change of variables formula holds: for any function $g : E \rightarrow F$ (where F is any complete real vector space), then

$$\int_{y \in f(s)} g(y) \, d\mu(y) = \int_{x \in s} |\det(f'(x))| g(x) \, d\mu(x).$$

This corresponds perfectly to Theorem 3.2.

Let us list the different domains of mathematics that are involved in the statement of Theorem 3.2:

1. Analysis and topology: to talk about normed spaces and continuity.
2. Calculus: to make sense of derivatives.
3. Measure theory: to talk about integrals and measures of sets (including the definition of additive Haar measures).
4. Linear algebra: to talk about finite dimensional spaces, and also about determinants of linear maps.

All these domains should be formalized before the above formalized statement `integral_image_eq_integral_abs_det_fderiv_smul` can be merely written down and understood by the system.

There are also tools that show up in the proof of the theorem, but not its statement:

5. Ordinals and transfinite induction (these show up in the proof of the covering theorems).
6. Linear maps rescale Lebesgue measures according to the absolute value of their determinants.
7. Covering theorems, like the Besicovitch and Vitali covering theorems.
8. Descriptive set theory, notably the theory of Polish spaces and analytic sets in them (they are instrumental in the proof of the Lusin-Souslin theorem).

All these should be formalized before the proof of the change of variables theorem.

This project resulted in 80 pull requests to `mathlib`, adding roughly 15,000 lines of codes. Among these, most are devoted to the prerequisites presented above: the file on the change of variables formula itself has only 1259 lines, less than 10% of the total. In the topics above, Items 1–5 were already mature enough that they needed few additions. Items 6–8 form the bulk of the formalization of this project, with roughly 20% for 6, the remaining 70% being split evenly between 7 and 8.

`mathlib` is an open source project: everyone can submit pull requests, which are then submitted to a thorough refereeing process. There are 25 maintainers of the project. When a maintainer is happy with a pull request (and several sanity checks have been automatically performed, as explained in [2]), then he can merge it to the main branch (and of course no maintainer can merge his own pull requests). Given the width of `mathlib`, no maintainer is expert in all areas: the pull requests in this project were therefore refereed by different maintainers depending on their domains. An important point is that the maintainers coordinate to ensure the unity of the whole library. For instance, the linear maps that are used in linear algebra are the same as those that are used in algebraic applications such as Galois theory, or in analytic applications such as derivatives of maps.

This inter-operability is extremely useful for a project such as the change of variables formula, that involves many different areas of mathematics: in a less coherent project, one would likely need to add glue to make sure that different modules can work together, and this would become quickly unwieldy at this level of complexity. In this respect, in the language of [10], the `mathlib` library

is cathedral-like as it is complex and coherent, but its open-source development process also has some bazaar characteristics. The delicate balance between these two models is only possible thanks to the hard work of the maintainers, who should be thanked for their dedication.

The next three sections will be devoted to more in-depth discussion of the three main ingredients 6–8. Before that, let us make a few remarks on the formalized statement of the theorem.

Remark 4.1. There are several assumptions in this theorem that appear between brackets, like `[normed_group E]`. These are *typeclass assumptions*, that should be filled automatically by the system when the theorem is used. The only assumptions that should be checked by the user are those between parentheses, like `(hs : measurable_set s)`. The typeclass assumptions are checked by the system using special lemmas that are tagged as *instances*. For instance, the fact that the volume on \mathbb{R} is a Lebesgue measure is an instance, as well as the fact that the product of two Lebesgue measures is a Lebesgue measure, so the theorem will automatically apply to the standard Lebesgue measure on $E = \mathbb{R} \times \mathbb{R}$ (and the fact that $\mathbb{R} \times \mathbb{R}$ is a finite-dimensional real normed vector space is also checked automatically by typeclass inference).

Remark 4.2. In this version of the theorem, E is a general finite-dimensional real vector space, and μ is a general Lebesgue measure on E (this is the content of the typeclass assumption `[is_add_haar_measure μ]`). One may wonder if one really needs this generality, and if it would not be more natural to have the theorem only on \mathbb{R}^n with its canonical (product) Lebesgue measure. For instance, this is the way the analogue of this theorem is formulated in Isabelle/HOL. However, we found that this more restricted version is too limited.

As an illustration, let us recall a classical proof of the value of the Gaussian integral $\int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}$. Denoting by I the value of the integral, one can compute using a polar change of coordinates in $\mathbb{R} \times \mathbb{R}$, writing $(x, y) = (r \cos \theta, r \sin \theta)$ as follows:

$$\begin{aligned} I^2 &= \int_{\mathbb{R} \times \mathbb{R}} e^{-(x^2+y^2)/2} dx dy = \int_{r=0}^{+\infty} \int_{\theta=-\pi}^{\pi} e^{-r^2/2} \cdot r dr d\theta \\ &= 2\pi \int_{r=0}^{\infty} r e^{-r^2/2} dr = 2\pi, \end{aligned}$$

where the computation $\int_{r=0}^{\infty} r e^{-r^2/2} dr = 1$ follows from the fact that $r e^{-r^2/2}$ is the derivative of $e^{-r^2/2}$.

This proof (which has been formalized in `mathlib`) uses the change of variables theorem in the space $E = \mathbb{R} \times \mathbb{R}$, with the product Lebesgue measure. But this space is *not* one of the spaces \mathbb{R}^n , defined as the space of functions from $\{0, \dots, n-1\}$ to \mathbb{R} . One could try to reformulate the above proof using \mathbb{R}^2 instead of $\mathbb{R} \times \mathbb{R}$, but then Fubini theorem (which has been used in the first step of the above proof) would not apply directly as it only makes sense for product spaces.

This is an important lesson: *more general theorems apply in more situations, so one should aim for generality in formalized mathematics library to improve their usability.*

Remark 4.3. The differentiability assumption inside a set is not completely standard in mathematics, but it shows up often in particular cases. Notably, in one dimension, one often talks about left derivatives and right derivatives, which are particular cases of Definition 3.1 where s is respectively $(-\infty, x]$ or $[x, +\infty)$. In differential geometry, one also often encounters functions which are differentiable on half-spaces or on submanifolds. With these examples in mind, derivatives in `mathlib` were defined from the start using Definition 3.1, so they were already general enough for Theorem 3.2. As most of the beginnings of analysis and topology in `mathlib`, this is strongly inspired from the Isabelle/HOL formalization of analysis described in [7].

Remark 4.4. There is a difference between the statement of Theorem 3.2 and its formalized version: in Theorem 3.2, we require the function g to be integrable on $f(s)$, but this assumption is nowhere to be seen in the formalized version (not even measurability or almost everywhere measurability of g). This makes the formalized version easier to use: the user does not need to worry about proving the integrability of the function.

The formalized version comes with a companion theorem, saying that g is integrable on $f(s)$ if and only if $x \mapsto |\det f'(x)| \cdot g(x)$ is integrable on s , and called `integrable_on_image_iff_integrable_on_abs_det_fderiv_smul`. Since the integral of non-integrable functions is defined to be 0 by convention, the theorem is then tautologically true for non-integrable functions as both sides of the statement vanish.

The companion theorem is not trivial, especially regarding measurability issues. Ultimately, it relies on the fact that the restriction of f to s is a measurable embedding, i.e., it maps measurable sets to measurable sets. This follows from the deep Lusin-Souslin theorem in descriptive set theory. A first version of the formalization had the additional assumption that g was integrable and avoided the Lusin-Souslin theorem. It was less satisfactory since it required more work from the end user of the theorem when applying it.

This rule is followed throughout `mathlib`: *One should try to minimize the assumptions of theorems to make them easier to apply for the users, even if this comes with a higher proof burden for the formalizer of the theorem.* (In our specific case, the Lusin-Souslin theorem indeed required several thousands additional lines of formalization.)

Let us give another silly example of this rule: the formula $\int f + g = \int f + \int g$ requires the functions f and g to be integrable. On the other hand, the formula $\int cf = c \int f$ is true whether f is integrable or not, so the latter formula should be given without integrability assumptions, even though the proof becomes more complicated as it requires several case distinctions. Indeed, if f is nonintegrable and c is nonzero, then cf is also nonintegrable so both sides vanish; if f is nonintegrable and c is zero, then cf becomes integrable as it is the zero function,

but both sides vanish again; if f is integrable then cf is also integrable and one is back to the usual situation.

5 Linear maps rescale Lebesgue measure according to their determinants

A basic ingredient of Theorem 3.2 is that it holds at an infinitesimal level, i.e., for the linearized map: a linear map should act on Lebesgue measure by multiplying it according to the absolute value of its determinant.

Here is the formalized statement:

```
lemma add_haar_image_linear_map
  [normed_group E] [normed_space ℝ E] [measurable_space E] [borel_space E]
  [finite_dimensional ℝ E] (μ : measure E) [is_add_haar_measure μ]
  (f : E →ₗ[ℝ] E) (s : set E) :
  μ (f '' s) = ennreal.of_real (abs f.det) * μ s
```

The proof of this theorem splits into two steps.

Let us first prove it on $E = \mathbb{R}^n$, with its standard product measure. A linear map f on this space corresponds in a canonical way to a matrix M . Gaussian elimination shows that M can be written as the product of transvection matrices (i.e., matrices with ones on the diagonal and at most one non-zero off-diagonal coefficient) and a diagonal matrix. As the statement to be proved is clearly stable under multiplication (as the determinant of a product is the product of the determinants), it therefore suffices to check it for transvections and for diagonal matrices. For diagonal matrices, it follows readily from the fact that the measure is a product measure and from the elementary 1-dimensional situation. For transvections (whose determinant is 1), we should show that a transvection preserves Lebesgue measure. This follows from a straightforward computation using Fubini to single out the coordinate at which there is a nonzero entry in the matrix.

The key argument of this step is thus Gaussian elimination, which was not yet proved in `mathlib` before this project and was formalized with this goal in mind. This is probably the most unexpected outcome of this project!

Let us now turn to the case of a general finite-dimensional vector space E , with a general Lebesgue measure μ . There is no canonical basis, and therefore no way to identify a linear map with a matrix, and moreover a general Lebesgue measure has no product structure a priori, so that the above argument does not make sense. However, one can choose a basis, which yields an isomorphism A between \mathbb{R}^n and E for some n . Let $g := A^{-1} \circ f \circ A$ be the conjugate of f under this isomorphism. Its determinant coincides with that of f , and moreover the first step applies to g . To conclude, it suffices to show that the image under A of the standard Lebesgue measure on \mathbb{R}^n coincides with μ (or a scalar multiple of μ). We deduce this from uniqueness of Lebesgue measure up to scalar multiplication, which was already available in `mathlib` in the right generality: two Haar measures on a locally compact group are multiples of each other. This is

a nontrivial fact, which had fortunately already been formalized by van Doorn with a totally unrelated application in mind, see [1].

Remark 5.1. Once the above theorem is available, one deduces that the volume of balls behaves like $\mu(B(x, r)) = r^d \mu(B(0, 1))$ where d is the dimension of the space. This fact makes it possible to apply the Vitali covering theorem (Theorem 6.1 below) to μ , and is therefore fundamental. Note that it is obvious for the standard product measure in the Euclidean space \mathbb{R}^n , but not so obvious in a general normed vector space with a general Lebesgue measure.

6 Covering theorems in measure theory

There is a huge variety of covering theorems for measure theory in the literature. Among them, the two prominent most ones are probably the Vitali and Besicovitch covering theorems that we will now describe and that we have formalized for the current project.

Here is a version of the Vitali covering theorem:

Theorem 6.1. *In a metric space, consider a locally finite measure μ which is doubling: there exists $C > 0$ such that, for any x, r , then $\mu(B(x, 2r)) \leq C\mu(B(x, r))$. Consider a set of balls t with uniformly bounded radii, and a set s at which the family t is fine, i.e., every point of s belongs to balls in t with arbitrarily small radius. Then there exists a disjoint subfamily of t covering almost all s .*

Here is a version of the Besicovitch covering theorem:

Theorem 6.2. *Let E be a finite-dimensional real normed vector space, and s a subset of E . Consider a set t of balls such that, for any $x \in s$, there exist arbitrarily small radii such that $B(x, r) \in t$. Let μ be a sigma-finite measure. Then there exists a disjoint subfamily of t covering almost all s .*

The assumptions of the Besicovitch theorem on the measure are weaker than those of the Vitali theorem, as they do not assume any doubling property. On the other hand, the former theorem requires stronger geometric assumptions (in this formulation, a finite-dimensional real normed vector space instead of a general metric space). The Besicovitch theorem is harder to prove and often more powerful than the Vitali theorem. The book [3] is a standard reference for these theorems and several powerful extensions, which was used for the formalization.

Both theorems are consequences of purely combinatorial results, from which the measure theoretic versions are then deduced. For instance, the deterministic result leading to the Besicovitch covering theorem is the following statement:

Theorem 6.3. *Let E be a finite-dimensional real normed vector space. There exists a constant $N = N(E)$ with the following property. Consider a set s , and a set of balls t with uniformly bounded radii such that any point in s is the center of some ball in t . Then one may find disjoint subfamilies t_1, \dots, t_N of t which still cover s .*

To deduce the measure-theoretic version from this deterministic version, one picks one of the subfamilies covering a proportion $1/N$ of s , say t_i , and then works inductively on the subset of s that it still to be covered.

The best value of the constant N has been determined in [6]: it is the maximal number of points one can put inside the unit ball of radius 2 under the condition that their distances are bounded below by 1. This is the version we have formalized in `mathlib`.

The book [3] introduces a formalism to deduce consequences of covering theorems in a uniform way, independently of the covering theorem. This formalism is often considered to be too abstract by mathematicians, but it turns out to be extremely well suited to formalization. Let us say that a family of sets $(\mathcal{F}_x)_{x \in X}$ in a metric measured space X is a Vitali family if it satisfies the following property: consider a (possibly non-measurable) set s , and for any x in s a subfamily F_x of \mathcal{F}_x containing sets of arbitrarily small diameter. Then one can extract from $\bigcup_{x \in s} F_x$ a disjoint subfamily covering almost all s .

In this language, the Vitali covering theorem says that one gets a Vitali family by taking for \mathcal{F}_x the balls that contain x , in a space where the measure μ is doubling. And the Besicovitch covering theorem states that, in a finite-dimensional real vector space, one gets a Vitali family by taking for \mathcal{F}_x the balls centered at x .

The fundamental theorem on differentiation of measures is the following. On a metric space with a measure μ , consider a Vitali family \mathcal{F} and another measure ρ . Then, for μ -almost every x , the ratio $\rho(a)/\mu(a)$ converges when a shrinks to x along the Vitali family \mathcal{F}_x , towards the Radon-Nikodym derivative of ρ with respect to μ . We have formalized this theorem, as follows:

```

theorem vitali_family.ae_tendsto_rn_deriv
  [sigma_compact_space  $\alpha$ ] [borel_space  $\alpha$ ]
  { $\mu$  : measure  $\alpha$ } [is_locally_finite_measure  $\mu$ ] (v : vitali_family  $\mu$ )
  ( $\rho$  : measure  $\alpha$ ) [is_locally_finite_measure  $\rho$ ] :
   $\forall^m x \partial\mu$ , tendsto ( $\lambda a, \rho a / \mu a$ ) (v.filter_at x) (nhds ( $\rho$ .rn_deriv  $\mu$  x))

```

Note that the convergence when a set in a Vitali family shrinks to a point (i.e., its diameter tends to 0) is not one of the standard convergences in mathematics (and Federer discusses it at length and introduces a special notation for it) but it fits perfectly well within the framework of `mathlib` where all notions of convergence are expressed with the single notion of filter, as advocated by [7].

Once this theorem is formalized, versions in the context of the Vitali and the Besicovitch covering theorems readily follow. In these respective contexts, they make it possible to prove that almost every point x of a set s is a Lebesgue density point of s , i.e., $\mu(s \cap B(x, r))/\mu(s) \rightarrow 1$ as $r \rightarrow 0$. This fact plays a key role in the proof of Theorem 3.2 to prove that $x \mapsto f'(x)$ is almost everywhere measurable (indeed, it is measurable when restricted to the set of Lebesgue density points).

7 Polish spaces and descriptive set theory

A Polish space is a topological space which is second-countable and on which there exists a complete metric space structure inducing the given topology. A good reference on Polish spaces and descriptive set theory is [8]. This definition may seem strange at first: one could instead require to have a complete second-countable metric space. However, in many applications, there is no natural distance, and the only relevant information is the topology. For instance, the extended reals $[-\infty, \infty]$ are a Polish space, but they don't have a canonical metric to work with.

This slightly awkward definition makes Polish spaces slightly awkward to use in proof assistants, as it refers to the mere existence of a nice metric but without providing it. In the completely classical framework of `mathlib`, this is not a real issue as one can use choice to pick such an arbitrary nice metric. The definition is formalized as follows:

```
class polish_space (α : Type*) [h : topological_space α] : Prop :=
(second_countable [] : second_countable_topology α)
(complete : ∃ m : metric_space α,
  m.to_uniform_space.to_topological_space = h ∧
  @complete_space α m.to_uniform_space)
```

To construct a nice metric space structure on a Polish space, one uses the following incantation in proofs: `letI := upgrade_polish_space α`. It endows the Polish space α with a metric space structure which is registered in the typeclass system as complete and second-countable, and moreover the topology associated to this metric is definitionally equal to the given topology. This makes it possible to work smoothly with Polish spaces as one would do in a non-formalized setting.

An important theme in descriptive set theory is to start with a Polish space and modify its topology to get better behavior while retaining Polishness. For instance, if a map between two Polish spaces is measurable, then one can construct a finer Polish topology on the source space for which the map becomes continuous. This makes it possible to deduce results for measurable maps from results for continuous maps. Unfortunately, this kind of argument does not interact well with the typeclass system, in which each type is supposed to be endowed with at most one typeclass of each kind, and in particular at most one topology. Fortunately, there is a way to override typeclass inference and provide explicitly the topology one would like to use (at the cost of added verbosity and reduced readability). For instance, let us give the formalized version of the following basic (but nontrivial) statement: given a Polish space and countably many finer Polish topologies, there exists another Polish topology which is finer than all of them.

```
lemma exists_polish_space_forall_le {ι : Type*} [encodable ι]
[t : topological_space α] [p : polish_space α]
(m : ι → topological_space α) (hm : ∀ n, m n ≤ t)
(h'm : ∀ n, @polish_space α (m n)) : ∃ (t' : topological_space α),
  (∀ n, t' ≤ m n) ∧ (t' ≤ t) ∧ @polish_space α t' :=
```

This fact is proved as follows. First, one checks (by constructing a suitable complete metric) that a countable product of Polish spaces is Polish. Consider then the diagonal embedding of α into the infinite product $Y = \alpha^{\mathbb{N}}$, where the n -th copy of α is endowed with the n -th topology m_n . Then Y is Polish, therefore the pullback of its topology under the diagonal embedding is also Polish and it satisfies all the required properties.

Another trick that proves useful in this kind of argument is to use a type synonym, i.e., a copy of a type with a different name. As typeclass inference does not unfold definitions, the type synonym is not endowed with any topology or metric, and one can register new instances that will not conflict with the original ones. As an example, let us sketch the proof that an open subset of a Polish space is Polish:

```
lemma is_open.polish_space [topological_space  $\alpha$ ] [polish_space  $\alpha$ ]
{s : set  $\alpha$ } (hs : is_open s) : polish_space s
```

One endows α with a distance for which it is complete and second-countable. Then the open subset s of α has the induced topology, which is second-countable, and it also inherits the restricted distance. But in general it is not complete for the restricted distance (think of the interval $(0, 1) \subset \mathbb{R}$). One should therefore add another distance. A suitable formula for it is

$$d'(x, y) = d(x, y) + |1/d(x, \alpha \setminus s) - 1/d(y, \alpha \setminus s)|.$$

As it blows up close to the boundary of s , one can check that this is a complete distance on s , defining the same topology as the original topology and therefore proving that it is Polish. For formalization purposes, we do not put this distance on s (which would then have two competing metric space structures), but instead on a type synonym `complete_copy s`. Then there is no difficulty to check that `complete_copy s` is Polish. As this new distance defines the same topology as the original one, the identity from s to `complete_copy s` is a homeomorphism, hence the fact that s itself is Polish follows.

The Lusin-Souslin theorem states that the image of a measurable set under a measurable injective map on a Polish space is still measurable. Here is the formalized statement:

```
theorem measurable_set.image_of_measurable_inj_on
[topological_space  $\gamma$ ] [polish_space  $\gamma$ ] [measurable_space  $\gamma$ ]
[borel_space  $\gamma$ ] [topological_space  $\beta$ ] [t2_space  $\beta$ ] [measurable_space  $\beta$ ]
[borel_space  $\beta$ ] [second_countable_topology  $\beta$ ] {s : set  $\gamma$ } {f :  $\gamma \rightarrow \beta$ }
(hs : measurable_set s) (f_meas : measurable f) (f_inj : set.inj_on f s) :
measurable_set (f '' s)
```

Its proof builds on the techniques we have sketched above, but it is considerably more involved. We will not get into the details of the proof. Let us just mention that it is first proved assuming that f is continuous, and then generalized to a measurable f by using the trick to modify the topology to turn a measurable map into a continuous map. A key notion in the proof is that of

an analytic set, i.e., the image of a Polish space under a continuous map, and a key result is that if two analytic sets are disjoint, then they are contained in disjoint measurable sets (this result is called the Lusin separation theorem). As far as the author knows, there is no known proof of the Lusin-Souslin theorem which does not go through a study of analytic sets, even though these sets do not appear in the conclusion of the theorem.

8 Conclusion

We have described the formalization of a not-so-deep theorem of mathematics, the change of variables formula in integrals, but in an advanced version. One interest of this theorem from the point of view of formalization is that it involves several areas of mathematics that are often considered quite independent, but that need to interact seamlessly here – as is often the case in advanced mathematics that mix basic results from several areas.

We have explained how the development model of `mathlib` has made this project reasonable, as well as its general philosophy: things should be done right, in the greatest level of generality, and without taking shortcuts. A lot of refactors are needed to reach this goal. This is possible in `mathlib` since it is a mono-repository project, without a lot of outside users, and would be harder for more mature projects. We may hope that basic definitions stabilize with time, but we are clearly not there yet: the definition of a group was changed less than one year ago to cope with a definitional equality issue in tensor products of abelian groups seen as \mathbb{Z} -modules, that showed up when doing advanced mathematics in the liquid tensor experiment. This kind of agile development is clearly a strength of `mathlib` currently, but once it reaches a critical size other strategies will need to be devised to make sure it can be used in a more stable way by other projects.

The change of variables formula is one of the tools to implement de Rham cohomology. While this project was being done, other necessary tools have been formalized independently, for other projects: homological algebra was developed for the liquid tensor experiment, and vector bundles on manifolds were developed for the sphere eversion project. This means that de Rham cohomology is now in sight.

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