1. Equation 1722

I work with the free multiplicative group G on countably many generators as usual. The functional equation is

$$f(f(hf(h)^{-1})f(h)h^{-1}f(1)^{-1}) = h^{-1}f(1)^{-1}.$$

Think of this as saying that if $(1, x_1), (a, b), (ab^{-1}, c) \in f$, then $(cba^{-1}x_1^{-1}, a^{-1}x_1^{-1}) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

(1) E is finite.
(2) E is a function.
(3) $(1, x_1), (x_1^{-1}, x_2), (x_2, x_1^{-1}), (x_1^{-1}x_2^{-1}, x_3), (x_3x_2, 1) \in E$ (where x_1, x_2, x_3 are generators of G).
(4) If $(a, b), (ab^{-1}, c) \in E$, then $(cba^{-1}x_1^{-1}, a^{-1}x_1^{-1}) \in E$.
(5) If $(a, b), (c, d) \in E$ and $ab^{-1} = cd^{-1}$, then $(a, b) = (c, d)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a, b) \in E$ for some $b \in G$.

If $ab^{-1} \in \mathrm{dom}(E)$, then by condition (4) we are already done. So reduce to the case when $ab^{-1} \notin \mathrm{dom}(E)$. By (3), note that $a \neq x_1^{-1}$. Let c be any generator not appearing anywhere in E, and fix

$$E' := E \cup \{(ab^{-1}, c), (cba^{-1}x_1^{-1}, a^{-1}x_1^{-1}\}.$$

Conditions (1), (2), and (3) on E' are clear, as is condition (5) from the newness of c.

For condition (4), we already noted that $a \neq x_1^{-1}$. A case analysis now suffices (using (5) to show that we don't cause any problems with the old pairs in E).

Case 2: Assume $a \notin \mathrm{dom}(E)$. If there is some $(x, y) \in E$ with $a = xy^{-1}$, then use Case 1 to make the functional equation hold for x, and then a will belong to the domain of that extension, and we reduce to Case 1 again.

So, we may assume there is not such pair (x, y). Taking b to be any generator of G not appearing in E or in a, then $E \cup \{(a, b)\} \in \mathscr{E}$ and we again revert to Case 1. ☐

With the choice that $f(1) = x_1$, the functional equation for 2644 is

$$f(x_1^{-2}) = 1.$$

The initial seed

$$\{(1, x_1), (x_1^{-1}, x_2), (x_2, x_1^{-1}), (x_1^{-1}x_2^{-1}, x_3), (x_3x_2, 1), (x_1^{-2}, x_4)\}$$

works to contradict this equation.

The functional equation for 3050 is

$$f(x_1^{-1}f(x_1^{-1})^{-1}f(x_1^{-1}f(x_1^{-1})^{-1})^{-1}) = x_1^{-1}f(x_1^{-1})^{-1}f(x_1^{-1}f(x_1^{-1})^{-1})^{-1}.$$

The initial seed

$$\{(1, x_1), (x_1^{-1}, x_2), (x_2, x_1^{-1}), (x_1^{-1}x_2^{-1}, x_3), (x_3x_2, 1), (x_1^{-1}x_2^{-1}x_3^{-1}, x_4), (x_4x_3x_2, x_2)\}$$

works to contradict this equation.

The functional equation for 1832 is

$$f(x_1f(x_1)^{-1}) = f(x_1)^{-1}.$$

The initial seed

$$\{(1,x_1),(x_1^{-1},x_2),(x_2,x_1^{-1}),(x_1^{-1}x_2^{-1},x_3),(x_3x_2,1),(x_1,x_4),(x_1x_4^{-1},x_5),(x_5x_4,1)\}$$

works to contradict this equation.