A BLUEPRINT FOR THE FORMALIZATION OF SEYMOUR'S MATROID DECOMPOSITION THEOREM

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Abstract. This document is a blueprint for the formalization in Lean of the structural theory of regular matroids underlying Seymour's decomposition theorem. We present a modular account of regularity via totally unimodular representations, show that regularity is preserved under 1-, 2-, and 3-sums, and establish regularity for several special classes of matroids, including graphic, cographic, and the matroid R_{10}.

The blueprint records the logical structure of the proof, the precise dependencies between results, and their correspondence with Lean declarations. It is intended both as a guide for the ongoing formalization effort and as a human-readable reference for the organization of the proof.

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1. Introduction

Seymour's decomposition theorem provides a structural characterization of regular matroids by expressing them as iterated 1-, 2-, and 3-sums of graphic matroids, cographic matroids, and a single exceptional matroid R_{10}. This result lies at the intersection of matroid theory, linear optimization, and combinatorial geometry, and it plays a central role in the theory of totally unimodular matrices and polynomial-time algorithms. Throughout this blueprint, we primarily work with finite matroids. Several results extend to matroids of finite rank or to infinite matroids, but these generalizations are not pursued systematically here.

Our presentation of the structural theory of regular matroids closely follows the exposition and terminology of Truemper's monograph [1], which serves as a primary reference for the matroid

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theory and the matrix-based approach adopted throughout this blueprint. We thank Klaus Truemper for helpful correspondences about the regularity of the 3-sum.

The present document is a *blueprint* for the formalization of this theory in the Lean4 proof assistant. Rather than presenting a traditional mathematical exposition, the blueprint records the logical structure of the proof, isolates intermediate results into modular components, and tracks the precise dependencies between statements. Each definition, lemma, and theorem is intended to correspond to a Lean declaration, and many proofs are deferred to Lean and indicated as such.

The blueprint is organized into several thematic parts. We begin by developing the necessary background on totally unimodular matrices, pivoting operations, and vector matroids. We then prove that regularity is preserved under 1-, 2-, and 3-sums of matroids. Finally, we establish regularity for certain special matroids – graphic matroids, cographic matroids, and the matroid R_{10} – thereby completing the ingredients needed for Seymour's decomposition.

2. Preliminaries

2.1. Total Unimodularity.

Definition 1. Matrix is a function that takes a row index and returns a vector, which is a function that takes a column index and returns a value. The former aforementioned identity is definitional, the latter is syntactical. By abuse of notation $\left(R^Y\right)^X \equiv R^{X\times Y}$ we do not curry functions in this text. When a matrix happens to be finite (that is, both X and Y are finite) and its entries are numeric, we like to represent it by a table of numbers.

Definition 2. Let A be a square matrix over a commutative ring whose rows and columns are indexed by the integers $\{1,\dots,n\}$. The determinant of A is

$$\det A = \sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{i,\sigma(i)}\right),$$

where the sum is computed over all permutations $\sigma\in S_n$, $\operatorname{sgn}(\sigma)$ denotes the sign of permutation σ, and $a_{i,j}\in R$ is the element of A corresponding to the i-th row and the j-th column.

Definition 3. Let R be a commutative ring. We say that a matrix $A\in R^{X\times Y}$ is totally unimodular, or TU for short, if for every $k\in\mathbb{N}$, every (not necessarily contiguous) $k\times k$ submatrix T of A has $\det T\in\{0,\pm1\}$.

Lemma 4. Let A be a TU matrix. Suppose rows of A are multiplied by $\{0,\pm1\}$ factors. Then the resulting matrix A' is also TU.

Proof. We prove that A' is TU by Definition 3. To this end, let T' be a square submatrix of A'. Our goal is to show that $\det T'\in\{0,\pm1\}$. Let T be the submatrix of A that represents T' before pivoting. If some of the rows of T were multiplied by zeros, then T' contains zero rows, and hence $\det T'=0$. Otherwise, T' was obtained from T by multiplying certain rows by -1. Since T' has finitely many rows, the number of such multiplications is also finite. Since multiplying a row by -1 results in the determinant getting multiplied by -1, we get $\det T'=\pm\det T\in\{0,\pm1\}$ as desired. □

Lemma 5. Let A be a TU matrix. Suppose columns of A are multiplied by $\{0,\pm1\}$ factors. Then the resulting matrix A' is also TU.

Proof. Apply Lemma 4 to $A^\top$. □

Definition 6. Given $k\in\mathbb{N}$, we say that a matrix A is k-partially unimodular, or k-PU for short, if every (not necessarily contiguous, not necessarily injective) $k\times k$ submatrix T of A has $\det T\in\{0,\pm1\}$.

Lemma 7. A matrix A is TU if and only if A is k-PU for every $k\in\mathbb{N}$.

Proof. This follows from Definitions 3 and 6. □

2.2. Pivoting.

Definition 8. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. A long tableau pivot in A on (x, y) is the operation that maps A to the matrix A' where

$$\forall i \in X,\ \forall j \in Y,\ A'(i,j) = \begin{cases} \frac{A(i,j)}{A(x,y)}, & \text{if } i = x, \\ A(i,j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x. \end{cases}$$

Lemma 9. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then performing the long tableau pivot in A on (x, y) yields a TU matrix.

Proof. See implementation in Lean. □

Definition 10. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Perform the following sequence of operations.

(1) Adjoin the identity matrix $1 \in R^{X \times X}$ to A, resulting in the matrix $B = \begin{bmatrix} 1 & A \end{bmatrix} \in R^{X \times (X \oplus Y)}$.
(2) Perform a long tableau pivot in B on (x, y), and let C denote the result.
(3) Swap columns x and y in C, and let D be the resulting matrix.
(4) Finally, remove columns indexed by X from D, and let A' be the resulting matrix.

A short tableau pivot in A on (x, y) is the operation that maps A to the matrix A' defined above.

Lemma 11. Let $A \in R^{X \times Y}$ be a matrix and let $(x, y) \in X \times Y$ be such that $A(x, y) \neq 0$. Then the short tableau pivot in A on (x, y) maps A to A' with

$$\forall i \in X,\ \forall j \in Y,\ A'(i,j) = \begin{cases} \frac{1}{A(x,y)}, & \text{if } i = x \text{ and } j = y, \\ \frac{A(x,j)}{A(x,y)}, & \text{if } i = x \text{ and } j \neq y, \\ -\frac{A(i,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j = y, \\ A(i,j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}, & \text{if } i \neq x \text{ and } j \neq y. \end{cases}$$

Proof. Follows by direct calculation. □

Lemma 12. Let $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{Q}^{\{X_1 \cup X_2\} \times \{Y_1 \times Y_2\}}$. Let $B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$ be the result of performing a short tableau pivot on $(x, y) \in X_1 \times Y_1$ in B. Then $B'_{12} = 0$, $B'_{22} = B_{22}$, and $\begin{bmatrix} B'_{11} \\ B'_{21} \end{bmatrix}$ is the matrix resulting from performing a short tableau pivot on (x, y) in $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$.

Proof. This follows by a direct calculation. Indeed, because of the 0 block in B, B_{12} and B_{22} remain unchanged, and since $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is a submatrix of B containing the pivot element, performing a short tableau pivot in it is equivalent to performing a short tableau pivot in B and then taking the corresponding submatrix. □

Lemma 13. Let $k \in \mathbb{N}$, let $A \in \mathbb{Q}^{k \times k}$, and let A' be the result of performing a short tableau pivot in A on (x, y) with $x, y \in \{1, \dots, k\}$ such that $A(x, y) \neq 0$. Then A' contains a submatrix A'' of size $(k-1) \times (k-1)$ with $|\det A''| = |\det A| / |A(x, y)|$.

Proof. Let $X = \{1, \dots, k\} \setminus \{x\}$ and $Y = \{1, \dots, k\} \setminus \{y\}$, and let $A'' = A'(X, Y)$. Since A'' does not contain the pivot row or the pivot column, $\forall (i, j) \in X \times Y$ we have $A''(i, j) = A(i,j) - \frac{A(i,y) \cdot A(x,j)}{A(x,y)}$. For $\forall j \in Y$, let B_j be the matrix obtained from A by removing row x and column j, and let B''_j be the matrix obtained from A'' by replacing column j with $A(X, y)$ (i.e., the pivot column without the pivot element). The cofactor expansion along row x in A yields

$$\det A = \sum_{j=1}^{k} (-1)^{y+j} \cdot A(x, j) \cdot \det B_j.$$

By reordering columns of every B_j to match their order in B''_j, we get

$$\det A = (-1)^{x+y} \cdot \left(A(x,y) \cdot \det A' - \sum_{j \in Y} A(x,j) \cdot \det B''_j \right).$$

By linearity of the determinant applied to $\det A''$, we have

$$\det A'' = \det A' - \sum_{j \in Y} \frac{A(x,j)}{A(x,y)} \cdot \det B''_j$$

Therefore, $|\det A''| = |\det A| / |A(x,y)|$. □

Lemma 14. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix and let $(x,y) \in X \times Y$ be such that $A(x,y) \neq 0$. Then performing the short tableau pivot in A on (x,y) yields a TU matrix.

Proof. See implementation in Lean, which uses Lemma 9. □

2.3. **Vector Matroids.**

Definition 15. A matroid M is a pair $(E, \mathcal{I})$ where E is a (possibly infinite) set and $\mathcal{I} \in 2^E$ is such that:

(1) $\emptyset \in \mathcal{I}$
(2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $I \in \mathcal{I}$.
(3) If $I \in \mathcal{I}$ is not maximal (with respect to set inclusion) and $B \in \mathcal{I}$ is maximal, then there exists an $x \in B \setminus I$ such that $I \cup \{x\} \in \mathcal{I}$.
(4) If $X \subseteq E$ and $I \subseteq X$ is such that $I \in \mathcal{I}$, then there exists an $J \in \mathcal{I}$ with $I \subseteq J \subseteq X$ that is maximal with respect to set inclusion.

We call E the ground set of M and $\mathcal{I}$ the collection of independent sets in M. We say that $B \in \mathcal{I}$ is a base of M if B is maximal in $\mathcal{I}$.

Definition 16. Let R be a division ring, let X and Y be sets, and let $A \in R^{X \times Y}$ be a matrix. The vector matroid of A is the matroid $M = (Y, \mathcal{I})$ where a set $I \subset Y$ is independent in M if and only if the columns of A indexed by I are linearly independent.

Definition 17. Let R be a division ring, let X and Y be disjoint sets, and let $S \in R^{X \times Y}$ be a matrix. Let $A = \begin{bmatrix} 1 & S \end{bmatrix} \in R^{X \times (X \cup Y)}$ be the matrix obtained from S by adjoining the identity matrix as columns, and let M be the vector matroid of A. Then S is called the standard representation of M.

Lemma 18. Let $S \in R^{X \times Y}$ be a standard representation of a vector matroid M. Then X is a base in M.

Proof. See implementation in Lean. □

Lemma 19. Adding extra zero rows to a full representation matrix of a vector matroid does not change the matroid.

Proof. See implementation in Lean. □

Lemma 20. Let $A \in \mathbb{Q}^{X \times Y}$ be a TU matrix, let M be the vector matroid of A, and let B be a base of M. Then there exists a matrix $S \in \mathbb{Q}^{B \times (Y \setminus B)}$ such that S is TU and S is a standard representation of M.

Proof. See Lean implementation, which uses Lemmas 9 and 19. □

Definition 21. Let R be a magma containing zero. The support of matrix $A \in R^{X \times Y}$ is $A^{\#} \in \{0,1\}^{X \times Y}$ given by

$$\forall i \in X,\ \forall j \in Y,\ A^{\#}(i,j) = \begin{cases} 0, & \text{if } A(i,j) = 0, \\ 1, & \text{if } A(i,j) \neq 0. \end{cases}$$

Lemma 22. Transpose of a support matrix is equal to a support of the transposed matrix.

Proof. Definitional equality. □

Lemma 23. Submatrix of a support matrix is equal to a support matrix of the submatrix.

Proof. Definitional equality. □

Lemma 24. If A is a matrix over $\mathbb{Z}_2$, then $A^\# = A$.

Proof. Check elementwise equality. □

Lemma 25. If two standard representation matrices of the same matroid have the same base, then they have the same support.

Proof. See implementation in Lean. □

Lemma 26. A square matrix is invertible iff its determinant is invertible.

Proof. This result is proved in Mathlib. □

Lemma 27. Let A be a rational TU matrix with finite number of rows and finite number of columns. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See Lean implementation, which uses Lemmas 23 and 26. □

Lemma 28. Let A be a rational TU matrix with finite number of rows. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See Lean implementation, which uses Lemma 27. □

Lemma 29. Let A be a rational TU matrix. Its rows are linearly independent iff the rows of its support matrix are linearly independent.

Proof. See Lean implementation, which uses Lemma 28. □

Lemma 30. Let A be a TU matrix.

(1) If a matroid is represented by A, then it is also represented by $A^\#$.
(2) If a matroid is represented by $A^\#$, then it is also represented by A.

Proof. See Lean implementation, which uses Lemmas 22, 23, and 29. □

2.4. Regular Matroids.

Definition 31. A matroid M is regular if there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A.

Definition 32. We say that $A' \in \mathbb{Q}^{X \times Y}$ is a TU signing of $A \in \mathbb{Z}_2^{X \times Y}$ if A' is TU and

$$\forall i \in X,\ \forall j \in Y,\ |A'(i,j)| = A(i,j).$$

Lemma 33. Let $B \in \mathbb{Z}_2^{X \times Y}$ be a standard representation matrix of a matroid M. Then M is regular if and only if B has a TU signing.

Proof. Suppose that M is regular. By Definition 31, there exists a TU matrix $A \in \mathbb{Q}^{X \times Y}$ such that M is a vector matroid of A. By Lemma 18, X (the row set of B) is a base of M. By Lemma 20, A can be converted into a standard representation matrix $B' \in \mathbb{Q}^{X \times Y}$ of M such that B' is also TU. Since B' and B are both standard representations of M, by Lemma 25 the support matrices $(B')^\#$ and $B^\#$ are the same. Lemma 24 gives $B^\# = B$. Thus, B' is TU and $(B')^\# = B$, so B' is a TU signing of B.

Suppose that B has a TU signing $B' \in \mathbb{Q}^{X \times Y}$. Then $A = [1 \mid B']$ is TU, as it is obtained from B' by adjoining the identity matrix. Moreover, by Lemma 30, A represents the same matroid as $A^\# = [1 \mid B]$, which is M. Thus, A is a TU matrix representing M, so M is regular. □

3. Regularity of 1-Sum

Definition 34. Let R be a magma containing zero (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ be matrices where X_ℓ, Y_ℓ, X_r, Y_r are pairwise disjoint sets. The 1-sum $B = B_\ell \oplus_1 B_r$ of B_ℓ and B_r is

$$B = \begin{bmatrix} B_\ell & 0 \\ 0 & B_r \end{bmatrix} \in R^{(X_\ell \cup X_r) \times (Y_\ell \cup Y_r)}.$$

Definition 35. A matroid M is a 1-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 34.

Lemma 36. Let A be a square matrix of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Then $\det A = \det A_{11} \cdot \det A_{22}$.

Proof. This result is proved in Mathlib. □

Lemma 37. Let B_ℓ and B_r from Definition 34 be TU matrices (over $\mathbb{Q}$). Then $B = B_\ell \oplus_1 B_r$ is TU.

Proof. We prove that B is TU by Definition 3. To this end, let T be a square submatrix of B. Our goal is to show that $\det T \in \{0, \pm 1\}$.

Let T_ℓ and T_r denote the submatrices in the intersection of T with B_ℓ and B_r, respectively. Then T has the form

$$T = \begin{bmatrix} T_\ell & 0 \\ 0 & T_r \end{bmatrix}.$$

First, suppose that T_ℓ and T_r are square. Then $\det T = \det T_\ell \cdot \det T_r$ by Lemma 36. Moreover, $\det T_\ell, \det T_r \in \{0, \pm 1\}$, since T_ℓ and T_r are square submatrices of TU matrices B_ℓ and B_r, respectively. Thus, $\det T \in \{0, \pm 1\}$, as desired.

Without loss of generality we may assume that T_ℓ has fewer rows than columns. Otherwise we can transpose all matrices and use the same proof, since TUness and determinants are preserved under transposition. Thus, T can be represented in the form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix},$$

where T_{11} contains T_ℓ and some zero rows, T_{22} is a submatrix of T_r, and T_{12} contains the rest of the rows of T_r (not contained in T_{22}) and some zero rows. By Lemma 36, we have $\det T = \det T_{11} \cdot \det T_{22}$. Since T_{11} contains at least one zero row, $\det T_{11} = 0$. Thus, $\det T = 0 \in \{0, \pm 1\}$, as desired. □

Theorem 38. *Let M be a 1-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 35. Since M_ℓ and M_r are regular, by Lemma 33, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_1 B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 37, and a direct calculation shows that B' is a signing of B. Thus, M is regular by Lemma 33. □

4. Regularity of 2-Sum

Definition 39. Let R be a semiring (we will use $R = \mathbb{Z}_2$ and $R = \mathbb{Q}$). Let $B_\ell \in R^{X_\ell \times Y_\ell}$ and $B_r \in R^{X_r \times Y_r}$ where $X_\ell \cap X_r = \{x\}$, $Y_\ell \cap Y_r = \{y\}$, X_ℓ is disjoint with Y_ℓ and Y_r, and X_r is disjoint with Y_ℓ and Y_r. Additionally, let $A_\ell = B_\ell(X_\ell \setminus \{x\}, Y_\ell)$ and $A_r = B_r(X_r, Y_r \setminus \{y\})$, and suppose $r = B_\ell(x, Y_\ell) \neq 0$ and $c = B_r(X_r, y) \neq 0$. Then the 2-sum $B = B_\ell \oplus_{2,x,y} B_r$ of B_ℓ and B_r is defined as

$$B = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix} \quad \text{where} \quad D = c \otimes r.$$

Here $D \in R^{X_r \times Y_\ell}$, and the indexing is consistent everywhere.

Definition 40. A matroid M is a 2-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 39.

Lemma 41. Let B_ℓ and B_r from Definition 39 be TU matrices (over $\mathbb{Q}$). Then $C = \begin{bmatrix} D & A_r \end{bmatrix}$ is TU.

Proof. Since B_ℓ is TU, all its entries are in $\{0, \pm 1\}$. In particular, r is a $\{0, \pm 1\}$ vector. Therefore, every column of D is a copy of y, $-y$, or the zero column. Thus, C can be obtained from B_r by adjoining zero columns, duplicating the y column, and multiplying some columns by -1. Since all these operations preserve TUess and since B_r is TU, C is also TU. □

Lemma 42. Let B_ℓ and B_r be matrices from Definition 39. Let B'_ℓ and B' be the matrices obtained by performing a short tableau pivot on $(x_\ell, y_\ell) \in X_\ell \times Y_\ell$ in B_ℓ and B, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B_r$.

Proof. Let

$$B'_\ell = \begin{bmatrix} A'_\ell \\ r' \end{bmatrix}, \quad B' = \begin{bmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{bmatrix}$$

where the blocks have the same dimensions as in B_ℓ and B, respectively. By Lemma 12, $B'_{11} = A'_\ell$, $B'_{12} = 0$, and $B'_{22} = A_r$. Equality $B'_{21} = c \otimes r'$ can be verified via a direct calculation. Thus, $B' = B'_\ell \oplus_{2,x,y} B_r$. □

Lemma 43. Let B_ℓ and B_r from Definition 39 be TU matrices (over $\mathbb{Q}$). Then $B_\ell \oplus_{2,x,y} B_r$ is TU.

Proof. By Lemma 7, it suffices to show that $B_\ell \oplus_{2,x,y} B_r$ is k-PU for every $k \in \mathbb{N}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since all entries of $B_\ell \oplus_{2,x,y} B_r$ are in $\{0, \pm 1\}$ by construction.

Suppose that for some $k \in \mathbb{N}$ we know that for any TU matrices B'_ℓ and B'_r (from Definition 39) their 2-sum $B'_\ell \oplus_{2,x,y} B'_r$ is k-PU. Now, given TU matrices B_ℓ and B_r (from Definition 39), our goal is to show that $B = B_\ell \oplus_{2,x,y} B_r$ is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix T of B has $\det T \in \{0, \pm 1\}$.

First, suppose that T has no rows in X_ℓ. Then T is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by Lemma 41, so $\det T \in \{0, \pm 1\}$. Thus, we may assume that T contains a row $x_\ell \in X_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y_\ell$ such that $T(x_\ell, y_\ell) \neq 0$. Indeed, if $T(x_\ell, y) = 0$ for all y, then $\det T = 0$ and we are done, and $T(x_\ell, y) = 0$ holds whenever $y \in Y_r$.

Since B is 1-PU, all entries of T are in $\{0, \pm 1\}$, and hence $T(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 13, performing a short tableau pivot in T on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix T'' such that $|\det T| = |\det T''|$. Since T is a submatrix of B, matrix T'' is a submatrix of the matrix B' resulting from performing a short tableau pivot in B on the same entry (x_ℓ, y_ℓ). By Lemma 42, we have $B' = B'_\ell \oplus_{2,x,y} B_r$ where B'_ℓ is the result of performing a short tableau pivot in B_ℓ on (x_ℓ, y_ℓ). Since B_ℓ is TU, by Lemma 14, B'_ℓ is also TU. Thus, by the inductive hypothesis applied to T'' and $B'_\ell \oplus_{2,x,y} B_r$, we have $\det T'' \in \{0, \pm 1\}$. Since $|\det T| = |\det T''|$, we conclude that $\det T \in \{0, \pm 1\}$. □

Theorem 44. *Let M be a 2-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 40. Since M_ℓ and M_r are regular, by Lemma 33, B_ℓ and B_r have TU signings B'_ℓ and B'_r, respectively. Then $B' = B'_\ell \oplus_{2,x,y} B'_r$ is a TU signing of B. Indeed, B' is TU by Lemma 43, and a direct calculation verifies that B' is a signing of B. Thus, M is regular by Lemma 33. □

5. Regularity of 3-Sum

5.1. Definition.

Definition 45. Let X_ℓ, Y_ℓ, X_r, and Y_r be sets satisfying the following properties:

- $X_\ell \cap X_r = \{x_2, x_1, x_0\}$ for some distinct x_0, x_1, and x_2;
- $Y_\ell \cap Y_r = \{y_0, y_1, y_2\}$ for some distinct y_0, y_1, and y_2;
- X_ℓ is disjoint with Y_r; and
- Y_ℓ is disjoint with X_r.

Let $B_\ell \in \mathbb{Z}_2^{X_\ell \times Y_\ell}$ and $B_r \in \mathbb{Z}_2^{X_r \times Y_r}$ be matrices of the form

$$B_\ell = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{|c|}{A_\ell} & 0 \\ \cline{2-4} & 1 & 1 & 0 \\ \hline D_\ell & \multicolumn{2}{c|}{D_0} & 1 \\ \cline{4-4} & \multicolumn{2}{c|}{} & 1 \\ \hline \end{array} \quad \text{and} B_r = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 & \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 & A_r \\ \cline{1-3} \multicolumn{2}{|c|}{D_r} & \multicolumn{2}{c|}{} \\ \hline \end{array}$$

where D_0 is invertible. Then the 3-sum $B = B_\ell \oplus_3 B_r$ of B_ℓ and B_r is defined as

$$B = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{|c|}{A_\ell} & 0 \\ \cline{2-3} & 1 & 1 & 0 \\ \hline D_\ell & D_0 & 1 & \\ \cline{3-3} & & 1 & A_r \\ \cline{1-3} D_{\ell r} & D_r & & \\ \hline \end{array} \quad \text{where } D_{\ell r} = D_r \cdot (D_0)^{-1} \cdot D_\ell.$$

Here the indexing is consistent between all the matrices, $D_0 \in \mathbb{Z}_2^{\{x_1,x_0\}\times\{y_0,y_1\}}$, and the submatrix $\begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array}$ is indexed by $\{x_2, x_1, x_0\} \times \{y_0, y_1, y_2\}$ in B_ℓ, B_r, and B.

Definition 46. A matroid M is a 3-sum of matroids M_ℓ and M_r if there exist standard $\mathbb{Z}_2$ representation matrices B_ℓ, B_r, and B (for M_ℓ, M_r, and M, respectively) of the form given in Definition 45.

5.2. Canonical Signing.

Lemma 47. Let $D_0 \in \mathbb{Z}_2^{\{x_1,x_0\}\times\{y_0,y_1\}}$ be an invertible matrix. Then, up to reindexing of rows and columns, either $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Proof. Brute force. □

For the sake of simplicity of notation, going forward we assume that the submatrix D_0 in Definition 45 falls into one of the two special cases presented in Lemma 47.

Definition 48. We call $D_0' \in \mathbb{Q}^{\{x_1,x_0\}\times\{y_0,y_1\}}$ the canonical signing of $D_0 \in \mathbb{Z}_2^{\{x_1,x_0\}\times\{y_0,y_1\}}$ if

$$D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_0' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{or} \quad D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_0' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly, we call $S' \in \mathbb{Q}^{\{x_2,x_1,x_0\}\times\{y_0,y_1,y_2\}}$ the canonical signing of $S \in \mathbb{Z}_2^{\{x_2,x_1,x_0\}\times\{y_0,y_1,y_2\}}$ if

$$S = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array} \quad \text{and } S' = \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{D_0'} & 1 \\ \cline{3-3} \multicolumn{2}{|c|}{} & 1 \\ \hline \end{array}$$

To simplify notation, going forward we use D_0, D_0', S, and S' to refer to the matrices of the form above. Observe that the canonical signing S' of S (from Definition 48) is TU.

Lemma 49. Let Q be a TU signing of S (from Definition 48). Let $u \in \{0, \pm 1\}^{\{x_2, x_1, x_0\}}$, $v \in \{0, \pm 1\}^{\{y_0, y_1, y_2\}}$, and Q' be defined as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \end{cases}$$
$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \end{cases}$$
$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in \{x_2, x_1, x_0\},\ \forall j \in \{y_0, y_1, y_2\}.$$

Then $Q' = S'$ (from Definition 48).

Proof. Since Q is a TU signing of S and Q' is obtained from Q by multiplying rows and columns by ± 1 factors, Q' is also a TU signing of S. By construction, we have

$$\begin{aligned}
&Q'(x_2, y_0) = Q(x_2, y_0) \cdot 1 \cdot Q(x_2, y_0) = 1, \\
&Q'(x_2, y_1) = Q(x_2, y_1) \cdot 1 \cdot Q(x_2, y_1) = 1, \\
&Q'(x_2, y_2) = 0, \\
&Q'(x_0, y_0) = Q(x_0, y_0) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_0) = 1, \\
&Q'(x_0, y_1) = Q(x_0, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot Q(x_2, y_1), \\
&Q'(x_0, y_2) = Q(x_0, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1, \\
&Q'(x_1, y_0) = 0, \\
&Q'(x_1, y_1) = Q(x_1, y_1) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_1)), \\
&Q'(x_1, y_2) = Q(x_1, y_2) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2)) \cdot (Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2)) = 1.
\end{aligned}$$

Thus, it remains to show that $Q'(x_0, y_1) = S'(x_0, y_1)$ and $Q'(x_1, y_1) = S'(x_1, y_1)$.

Consider the entry $Q'(x_0, y_1)$. If $D_0(x_0, y_1) = 0$, then $Q'(x_0, y_1) = 0 = S'(x_0, y_1)$. Otherwise, we have $D_0(x_0, y_1) = 1$, and so $Q'(x_0, y_1) \in \{\pm 1\}$, as Q' is a signing of S. If $Q'(x_0, y_1) = -1$, then

$$\det Q'(\{x_0, x_2\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of Q'. Thus, $Q'(x_0, y_1) = 1 = S'(x_0, y_1)$.

Consider the entry $Q'(x_1, y_1)$. Since Q' is a signing of S, we have $Q'(x_1, y_1) \in \{\pm 1\}$. Consider two cases.

(1) Suppose that $D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = 1$, then $\det Q = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = -1 = S'(x_1, y_1)$.
(2) Suppose that $D_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $Q'(x_1, y_1) = -1$, then $\det Q(\{x_1, x_0\}, \{y_1, y_2\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\}$, which contradicts TUness of Q'. Thus, $Q'(x_1, y_1) = 1 = S'(x_1, y_1)$.

□

Definition 50. Let X and Y be sets with $\{x_2, x_1, x_0\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU matrix. Define $u \in \{0, \pm 1\}^X$, $v \in \{0, \pm 1\}^Y$, and Q' as follows:

$$u(i) = \begin{cases} Q(x_2, y_0) \cdot Q(x_0, y_0), & i = x_0, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2) \cdot Q(x_1, y_2), & i = x_1, \\ 1, & i = x_2, \\ 1, & i \in X \setminus \{x_2, x_1, x_0\}, \end{cases}$$

$$v(j) = \begin{cases} Q(x_2, y_0), & j = y_0, \\ Q(x_2, y_1), & j = y_1, \\ Q(x_2, y_0) \cdot Q(x_0, y_0) \cdot Q(x_0, y_2), & j = y_2, \\ 1, & j \in Y \setminus \{y_0, y_1, y_2\}, \end{cases}$$

$$Q'(i, j) = Q(i, j) \cdot u(i) \cdot v(j) \quad \forall i \in X, \ \forall j \in Y.$$

We call Q' the canonical re-signing of Q.

Lemma 51. Let X and Y be sets with $\{x_2, x_1, x_0\} \subseteq X$ and $\{y_0, y_1, y_2\} \subseteq Y$. Let $Q \in \mathbb{Q}^{X \times Y}$ be a TU signing of $Q_0 \in \mathbb{Z}_2^{X \times Y}$ such that $Q_0(\{x_2, x_1, x_0\}, \{y_0, y_1, y_2\}) = S$ (from Definition 48). Then the canonical re-signing Q' of Q (from Definition 50) is a TU signing of Q_0 and $Q'(\{x_2, x_1, x_0\}, \{y_0, y_1, y_2\}) = S'$ (from Definition 48).

Proof. Since Q is a TU signing of Q_0 and Q' is obtained from Q by multiplying some rows and columns by ± 1 factors, Q' is also a TU signing of Q_0. Equality $Q'(\{x_2, x_1, x_0\}, \{y_0, y_1, y_2\}) = S'$ follows from Lemma 49. □

Definition 52. Suppose that B_ℓ and B_r from Definition 45 have TU signings B'_ℓ and B'_r, respectively. Let B''_ℓ and B''_r be the canonical re-signings (from Definition 50) of B'_ℓ and B'_r, respectively. Let A''_ℓ, A''_r, D''_ℓ, D''_r, and D''_0 be blocks of B''_ℓ and B''_r analogous to blocks A_ℓ, A_r, D_ℓ, D_r, and D_0 of B_ℓ and B_r. The canonical signing B'' of B is defined as

$$B'' = \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{A''_\ell} & & 0 \\ & \begin{matrix} 1 & 1 \end{matrix} & 0 & \\ \hline D''_\ell & D''_0 & \begin{matrix} 1 \\ 1 \end{matrix} & A''_r \\ D''_{\ell r} & D''_r & & \end{array}\right] \quad \text{where } D''_{\ell r} = D''_r \cdot (D''_0)^{-1} \cdot D''_\ell.$$

Note that D''_0 is non-singular by construction, so $D''_{\ell r}$ and hence B'' are well-defined.

5.3. Properties of Canonical Signing.

Lemma 53. B'' from Definition 52 is a signing of B.

Proof. By Lemma 51, B''_ℓ and B''_r are TU signings of B_ℓ and B_r, respectively. As a result, blocks A''_ℓ, A''_r, D''_ℓ, D''_r, and D''_0 in B'' are signings of the corresponding blocks in B. Thus, it remains to show that $D''_{\ell r}$ is a signing of $D_{\ell r}$. This can be verified via a direct calculation. □

Lemma 54. Suppose that B_r from Definition 45 has a TU signing B'_r. Let B''_r be the canonical re-signing (from Definition 50) of B'_r. Let $c''_0 = B''_r(X_r, y_0)$, $c''_1 = B''_r(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Then the following statements hold.

(1) For every $i \in X_r$, $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \left\{ \begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix} \right\}$.
(2) For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.
(3) $\begin{bmatrix} c''_0 & c''_2 & A''_r \end{bmatrix}$ is TU.
(4) $\begin{bmatrix} c''_1 & c''_2 & A''_r \end{bmatrix}$ is TU.
(5) $\begin{bmatrix} c''_0 & c''_1 & c''_2 & A''_r \end{bmatrix}$ is TU.

Proof. Throughout the proof we use that B''_r is TU, which holds by Lemma 51.

(1) Since B''_r is TU, all its entries are in $\{0, \pm 1\}$, and in particular $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}}$. If $\begin{bmatrix} c'_0(i) & c''_1(i) \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$, then

$$\det B''_r(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2 \notin \{0, \pm 1\},$$

which contradicts TUness of B''_r. Similarly, if $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$, then

$$\det B''_r(\{x_2, i\}, \{y_0, y_1\}) = \det \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = 2 \notin \{0, \pm 1\},$$

which contradicts TUness of B''_r. Thus, the desired statement holds.

(2) Follows from item 1 and a direct calculation.

(3) Performing a short tableau pivot in B''_r on (x_2, y_0) yields:

$$B''_r = \begin{bmatrix} \boxed{1} & 1 & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ -c_0 & c''_1 - c_0 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $\begin{bmatrix} c''_0 & c''_2 & A''_r \end{bmatrix}$ by removing row x_2 and multiplying columns y_0 and y_1 by -1. Since B''_r is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, we conclude that $\begin{bmatrix} c''_0 & c''_2 & A''_r \end{bmatrix}$ is TU.

(4) Similar to item 4, performing a short tableau pivot in B''_r on (x_2, y_1) yields:

$$B''_r = \begin{bmatrix} 1 & \boxed{1} & 0 \\ c_0 & c_1 & A_r \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 0 \\ c''_0 - c_1 & -c_1 & A_r \end{bmatrix}$$

The resulting matrix can be transformed into $\begin{bmatrix} c''_1 & c''_2 & A''_r \end{bmatrix}$ by removing row x_2, multiplying column y_1 by -1, and swapping the order of columns y_0 and y_1. Since B''_r is TU and since TUness is preserved under pivoting, taking submatrices, multiplying columns by ± 1 factors, and re-ordering columns, we conclude that $\begin{bmatrix} c''_1 & c''_2 & A''_r \end{bmatrix}$ is TU.

(5) Let V be a square submatrix of $\begin{bmatrix} c''_0 & c''_1 & c''_2 & A''_r \end{bmatrix}$. Our goal is to show that $\det V \in \{0, \pm 1\}$.

Suppose that column c''_2 is not in V. Then V is a submatrix of B''_r, which is TU. Thus, $\det V \in \{0, \pm 1\}$. Going forward we assume that column z is in V.

Suppose that columns c''_0 and c''_1 are both in V. Then V contains columns c''_0, c''_1, and $c''_2 = c''_0 - c''_1$, which are linearly. Thus, $\det V = 0$. Going forward we assume that at least one of the columns c''_0 and c''_1 is not in V.

Suppose that column c''_1 is not in V. Then V is a submatrix of $\begin{bmatrix} c''_0 & c''_2 & A''_r \end{bmatrix}$, which is TU by item 3. Thus, $\det V \in \{0, \pm 1\}$. Similarly, if column c''_0 is not in V, then V is a submatrix of $\begin{bmatrix} c''_1 & c''_2 & A''_r \end{bmatrix}$, which is TU by item 4. Thus, $\det V \in \{0, \pm 1\}$.

□

Lemma 55. Suppose that B_ℓ from Definition 45 has a TU signing B'_ℓ. Let B''_ℓ be the canonical re-signing (from Definition 50) of B'_ℓ. Let $d''_0 = B''_\ell(x_0, Y_\ell)$, $d''_1 = B''_\ell(x_1, Y_\ell)$, and $d''_2 = d''_0 - d''_1$. Then the following statements hold.

(1) For every $j \in Y_\ell$, $\begin{bmatrix} d''_0(i) \\ d''_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_1, x_0\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(2) For every $j \in Y_\ell$, $d''_2(j) \in \{0, \pm 1\}$.

(3) $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_2 \end{bmatrix}$ is TU.

(4) $\begin{bmatrix} A''_\ell \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU.

(5) $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU.

Proof. Apply Lemma 54 to $B_\ell^\top$, or repeat the same arguments up to transposition. □

Lemma 56. Let B'' be from Definition 52. Let $c''_0 = B''(X_r, y_0)$, $c''_1 = B''(X_r, y_1)$, and $c''_2 = c''_0 - c''_1$. Similarly, let $d''_0 = B''(x_0, Y_\ell)$, $d''_1 = B''(x_1, Y_\ell)$, and $d''_2 = d''_0 - d''_1$. Then the following statements hold.

(1) For every $i \in X_r$, $c''_2(i) \in \{0, \pm 1\}$.
(2) If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_1 \otimes d''_1$. If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $D'' = c''_0 \otimes d''_0 - c''_0 \otimes d''_1 + c''_1 \otimes d''_1$.
(3) For every $j \in Y_\ell$, $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm c''_2\}$.
(4) For every $i \in X_r$, $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.
(5) $\begin{bmatrix} A''_\ell \\ D'' \end{bmatrix}$ is TU.

Proof.

(1) Holds by Lemma 54.2.
(2) Note that

$$\begin{bmatrix} D''_\ell \\ D''_{\ell r} \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_\ell, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot D''_0, \quad \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix}, \quad \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

Thus,

$$D'' = \begin{bmatrix} D''_\ell & D''_0 \\ D''_{\ell r} & D''_r \end{bmatrix} = \begin{bmatrix} D''_0 \\ D''_r \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} D''_\ell & D''_0 \end{bmatrix} = \begin{bmatrix} c''_0 & c''_1 \end{bmatrix} \cdot (D''_0)^{-1} \cdot \begin{bmatrix} d''_0 \\ d''_1 \end{bmatrix}.$$

Considering the two cases for D''_0 and performing the calculations yields the desired results.
(3) Let $j \in Y_\ell$. By Lemma 55.1, $\begin{bmatrix} d''_0(j) \\ d''_1(j) \end{bmatrix} \in \{0, \pm 1\}^{\{x_1, x_0\}} \setminus \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Consider two cases.
  (a) If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item 2 we have $D''(X_r, j) = d''_0(j) \cdot c''_0 + (-d''_1(j)) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm (c''_0 - c''_1)\}$.
  (b) If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item 2 we have $D''(X_r, j) = (d''_0(j) - d''_1(j)) \cdot c''_0 + d''_1(j) \cdot c''_1$. By considering all possible cases for $d''_0(j)$ and $d''_1(j)$, we conclude that $D''(X_r, j) \in \{0, \pm c''_0, \pm c''_1, \pm (c''_0 - c''_1)\}$.
(4) Let $i \in X_r$. By Lemma 54.1, $\begin{bmatrix} c''_0(i) & c''_1(i) \end{bmatrix} \in \{0, \pm 1\}^{\{y_0, y_1\}} \setminus \{\begin{bmatrix} 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \end{bmatrix}\}$. Consider two cases.
  (a) If $D''_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then by item 2 we have $D''(i, Y_\ell) = c''_0(i) \cdot d''_0 + (-c''_1(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.
  (b) If $D''_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then by item 2 we have $D''(i, Y_\ell) = c''_0(i) \cdot d''_0 + (c''_1(i) - c''_0(i)) \cdot d''_1$. By considering all possible cases for $c''_0(i)$ and $c''_1(i)$, we conclude that $D''(i, Y_\ell) \in \{0, \pm d''_0, \pm d''_1, \pm d''_2\}$.

(5) By Lemma 55.5, $\begin{bmatrix} A''_\ell \\ d''_0 \\ d''_1 \\ d''_2 \end{bmatrix}$ is TU. Since TUness is preserved under adjoining zero rows, copies of existing rows, and multiplying rows by ± 1 factors, $\begin{bmatrix} A''_\ell \\ 0 \\ \pm d''_0 \\ \pm d''_1 \\ \pm d''_2 \end{bmatrix}$ is also TU. By item 4, $\begin{bmatrix} A''_\ell \\ D'' \end{bmatrix}$ is a submatrix of the latter matrix, hence it is also TU.

□

5.4. **Proof of Regularity.**

Definition 57. Let X'_ℓ, Y'_ℓ, X'_r, Y'_r be sets and let x_0 and x_1 be distinct elements contained neither in X'_ℓ nor X'_r. Additionally, let $c_0, c_1 \in \mathbb{Q}^{X'_r \cup \{x_1, x_0\}}$ be column vectors. We define $\mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$ to be the family of matrices of the form $\begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ such that:

(1) $A_\ell \in \mathbb{Q}^{X'_\ell \times Y'_\ell}$, $A_r \in \mathbb{Q}^{(X'_r \cup \{x_1, x_0\}) \times Y'_r}$, and $D \in \mathbb{Q}^{(X'_r \cup \{x_1, x_0\}) \times Y'_\ell}$;
(2) $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU;
(3) for every $j \in Y'_\ell$, $D(X'_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$;
(4) $\begin{bmatrix} c_0 & c_1 & c_0 - c_1 & A_r \end{bmatrix}$ is TU;
(5) $\begin{bmatrix} A_\ell & 0 \\ D(x_0, Y'_\ell) & 1 \\ D(x_1, Y'_\ell) & 1 \end{bmatrix}$ is TU;
(6) $c_0(x_0) = 1$ and $c_0(x_1) = 0$;
(7) either $c_1(x_0) = 0$ and $c_1(x_1) = -1$, or $c_1(x_0) = 1$ and $c_1(x_1) = 1$.

Lemma 58. Let B'' be from Definition 52. Then $B'' \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c''_0, c''_1)$ with $X'_\ell = X_\ell \setminus \{x_1, x_0\}$, $X'_r = X_r \setminus \{x_2, x_1, x_0\}$, $Y'_\ell = Y_\ell \setminus \{y_2\}$, $Y'_r = Y_r \setminus \{y_0, y_1\}$, x_0 and x_1 are the same, $c''_0 = B''(X'_r, y_0)$, and $c''_1 = B''(X'_r, y_1)$.

Proof. Recall that $c''_0 - c''_1 \in \{0, \pm 1\}^{X'_r}$ by Lemma 56.1, so $\mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c''_0, c''_1)$ is well-defined. To see that $B'' \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c''_0, c''_1)$, note that all properties from Definition 57 are satisfied: property 3 holds by Lemma 56.3, property 4 holds by Lemma 54.5, and property 2 holds by Lemma 56.5. □

Lemma 59. Let $C \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$ from Definition 57. Let $x \in X'_\ell$ and $y \in Y'_\ell$ be such that $A_\ell(x, y) \neq 0$, and let C' be the result of performing a short tableau pivot in C on (x, y). Then $C' \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$.

Proof. Our goal is to show that C' satisfies all properties from Definition 57. Let $C' = \begin{bmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{bmatrix}$, and let $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ be the result of performing a short tableau pivot on (x, y) in $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$. Observe the following.

- By Lemma 12, $C'_{11} = A'_\ell$, $C'_{12} = 0$, $C'_{21} = D'$, and $C'_{22} = A_r$.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property 2 for C, all entries of A_ℓ are in $\{0, \pm 1\}$.
- $A_\ell(x, y) \in \{\pm 1\}$, as $A_\ell(x, y) \in \{0, \pm 1\}$ by the above observation and $A_\ell(x, y) \neq 0$ by the assumption.
- Since $\begin{bmatrix} A_\ell \\ D \end{bmatrix}$ is TU by property 2 for C, and since pivoting preserves TUness, $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ is also TU.

These observations immediately imply properties 4 and 2 for C'. Indeed, property 4 holds for C', since $C'_{22} = A_r$ and $\begin{bmatrix} c_0 & c_1 & c_0 - c_1 & A_r \end{bmatrix}$ is TU by property 4 for C. On the other hand, property 2 follows from $C'_{11} = A'_\ell$, $C'_{21} = D'$, and $\begin{bmatrix} A'_\ell \\ D' \end{bmatrix}$ being TU. Thus, it only remains to show that C' satisfies property 3. Let $j \in Y_r$. Our goal is to prove that $D'(X'_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Suppose $j = y$. By the pivot formula, $D'(X'_r, y) = -\frac{D(X'_r,y)}{A_\ell(x,y)}$. Since $D(X'_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property 3 for C and since $A_\ell(x, y) \in \{\pm 1\}$, we get $D'(X'_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$.

Now suppose $j \in Y_\ell \setminus \{y\}$. By the pivot formula, $D'(X'_r, j) = D(X'_r, j) - \frac{A_\ell(x,j)}{A_\ell(x,y)} \cdot D(X'_r, y)$. Here $D(X'_r, j),\ D(X'_r, y) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$ by property 3 for C, and $A_\ell(x, j) \in \{0, \pm 1\}$ and $A_\ell(x, y) \in \{\pm 1\}$ by the prior observations. Perform an exhaustive case distinction on $D(X'_r, j)$, $D(X'_r, y)$, $A_\ell(x, j)$, and $A_\ell(x, y)$. The number of cases can be significantly reduced by using symmetries. In every remaining case, we can either show that $D'(X'_r, j) \in \{0, \pm c_0, \pm c_1, \pm(c_0 - c_1)\}$, as desired, or obtain a contradiction with property 5. □

Lemma 60. Let $C \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$ from Definition 57. Then C is TU.

Proof. By Lemma 7, it suffices to show that C is k-PU for every $k \in \mathbb{N}$. We prove this claim by induction on k. The base case with $k = 1$ holds, since properties 4 and 2 in Definition 57 imply that A_ℓ, A_r, and D are TU, so all their entries of $C = \begin{bmatrix} A_\ell & 0 \\ D & A_r \end{bmatrix}$ are in $\{0, \pm 1\}$, as desired.

Suppose that for some $k \in \mathbb{N}$ we know that every $C' \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$ is k-PU. Our goal is to show that C is $(k+1)$-PU, i.e., that every $(k+1) \times (k+1)$ submatrix S of C has $\det V \in \{0, \pm 1\}$.

First, suppose that V has no rows in X'_ℓ. Then V is a submatrix of $\begin{bmatrix} D & A_r \end{bmatrix}$, which is TU by property 4 in Definition 57, so $\det V \in \{0, \pm 1\}$. Thus, we may assume that S contains a row $x_\ell \in X'_\ell$.

Next, note that without loss of generality we may assume that there exists $y_\ell \in Y'_\ell$ such that $V(x_\ell, y_\ell) \neq 0$. Indeed, if $V(x_\ell, y) = 0$ for all y, then $\det V = 0$ and we are done, and $V(x_\ell, y) = 0$ holds whenever $y \in Y'_r$.

Since C is 1-PU, all entries of V are in $\{0, \pm 1\}$, and hence $V(x_\ell, y_\ell) \in \{\pm 1\}$. Thus, by Lemma 13, performing a short tableau pivot in V on (x_ℓ, y_ℓ) yields a matrix that contains a $k \times k$ submatrix S'' such that $|\det V| = |\det V''|$. Since V is a submatrix of C, matrix V'' is a submatrix of the matrix C' resulting from performing a short tableau pivot in C on the same entry (x_ℓ, y_ℓ). By Lemma 59, we have $C' \in \mathcal{C}(X'_\ell, Y'_\ell, X'_r, Y'_r; c_0, c_1)$. Thus, by the inductive hypothesis applied to V'' and C', we have $\det V'' \in \{0, \pm 1\}$. Since $|\det V| = |\det V''|$, we conclude that $\det V \in \{0, \pm 1\}$. □

Lemma 61. B'' from Definition 52 is TU.

Proof. Combine the results of Lemmas 58 and 60. □

Theorem 62. *Let M be a 3-sum of regular matroids M_ℓ and M_r. Then M is also regular.*

Proof. Let B_ℓ, B_r, and B be standard $\mathbb{Z}_2$ representation matrices from Definition 46. Since M_ℓ and M_r are regular, by Lemma 33, B_ℓ and B_r have TU signings. Then the canonical signing B'' from Definition 52 is a TU signing of B. Indeed, B'' is a signing of B by Lemma 53, and B'' is TU by Lemma 61. Thus, M is regular by Lemma 33. □

6. Special Matroids

Definition 63. Let $A \in \mathbb{Q}^{X \times Y}$ be a matrix. If for all $j \in Y$, one has that $a_{i,j} = 0$ for all $i \in X$, or that there exists $i_1, i_2 \in X$ such that

$$a_{i,j} = \begin{cases} 1 & \text{if } i = i_1 \\ -1 & \text{if } i = i_2 \\ 0 & \text{otherwise,} \end{cases}$$

then we call A a node-incidence matrix for a (directed) graph whose nodes are indexed by X and whose edges are indexed by Y.

Definition 64. We say that a matroid is graphic if it can be represented by a node-incidence matrix.

Definition 65. Let S be a standard representation given by matrix B. The dual of S is given by $-B^\mathsf{T}$.

Definition 66. We say a matroid is co-graphic if its dual is graphic.

Definition 67. The matroid with standard representation

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

over $\mathbb{Z}_2$ is called R_{10}.

Theorem 68. *The matroid R_{10} is regular.*

Proof. See Lean implementation. □

Theorem 69. *Every graphic matroid is regular.*

Proof. See Lean implementation. □

6.1. **Regularity of Cographic Matroids.**

Lemma 70 (Row space of a standard representation)**.** Let X and Y be disjoint finite sets and let

$$B \in \mathbb{F}_2^{X \times Y}.$$

Consider the matrix

$$A := [\,\mathbf{1}_x \mid B\,] \in \mathbb{F}_2^{X \times (X \cup Y)},$$

where the columns are indexed by $E := X \cup Y$ and the rows by X. Then the row space of A is

$$\operatorname{row}(A) \;=\; \{\,(u,\, uB) \mid u \in \mathbb{F}_2^X\,\} \;\subseteq\; \mathbb{F}_2^X \oplus \mathbb{F}_2^Y \cong \mathbb{F}_2^E.$$

Proof. The x-th row of A is $(e_x, B_{x,*})$, where e_x is the standard basis vector in $\mathbb{F}_2^X$ and $B_{x,*}$ is the x-th row of B. A general linear combination of the rows is therefore

$$\sum_{x \in X} u_x (e_x, B_{x,*}) = \big(u,\, \sum_{x \in X} u_x B_{x,*}\big) = (u,\, uB),$$

where $u = (u_x)_{x \in X} \in \mathbb{F}_2^X$. Conversely, every pair (u, uB) arises in this way, so these are exactly the row vectors. □

Lemma 71 (Orthogonal complement of a standard row space)**.** Let $A = [\mathbf{1}_x \mid B]$ be as in Lemma 70, and let

$$U := \operatorname{row}(A) \subseteq \mathbb{F}_2^{X \cup Y}.$$

Then the orthogonal complement of U is

$$U^\perp = \{\,(bB^\mathsf{T},\, b) \mid b \in \mathbb{F}_2^Y\,\}.$$

Equivalently, if $B^* := -B^\mathsf{T}$, then

$$U^\perp = \{\,(bB^*,\, b) \mid b \in \mathbb{F}_2^Y\,\}.$$

Proof. Write vectors in $\mathbb{F}_2^{X\cup Y}$ as pairs (a,b) with $a \in \mathbb{F}_2^X$ and $b \in \mathbb{F}_2^Y$. By Lemma 70, any element of U has the form (u, uB) with $u \in \mathbb{F}_2^X$. The orthogonality condition $(a,b) \in U^\perp$ means

$$\begin{aligned} 0 &= (a,b)\cdot(u,uB) \\ &= a\cdot u + b\cdot(uB) \\ &= a\cdot u + (bB^\mathsf{T})\cdot u \\ &= (a + bB^\mathsf{T})\cdot u \end{aligned}$$

for all $u \in \mathbb{F}_2^X$. Hence we must have $a = bB^\mathsf{T}$, and then

$$U^\perp = \{\, (bB^\mathsf{T}, b) \mid b \in \mathbb{F}_2^Y \,\}.$$

Over $\mathbb{F}_2$ we have $-1 = 1$, so $B^* = -B^\mathsf{T} = B^\mathsf{T}$, yielding the alternative description. □

Lemma 72 (Row space of the dual standard matrix)**.** With B and $B^* = -B^\mathsf{T}$ as above, define

$$A^* := [\, \mathbf{1}_y \mid B^* \,] \in \mathbb{F}_2^{Y\times(X\cup Y)}.$$

Then

$$\mathrm{row}(A^*) = U^\perp,$$

where $U = \mathrm{row}(A)$ and $U^\perp$ is given by Lemma 71.

Proof. The y-th row of A^* is $(e_y, B^*_{y,*})$ with $e_y \in \mathbb{F}_2^Y$. A general linear combination of the rows is

$$\sum_{y\in Y} b_y(e_y, B^*_{y,*}) = \big(b,\, bB^*\big),$$

where $b = (b_y)_{y\in Y} \in \mathbb{F}_2^Y$. Thus

$$\mathrm{row}(A^*) = \{\, (b, bB^*) \mid b \in \mathbb{F}_2^Y \,\}.$$

Identifying $\mathbb{F}_2^{X\cup Y}$ as $\mathbb{F}_2^X \oplus \mathbb{F}_2^Y$ with coordinates ordered as (X, Y), this is exactly the set

$$\{\, (bB^*, b) \mid b \in \mathbb{F}_2^Y \,\},$$

which coincides with $U^\perp$ by Lemma 71. □

Lemma 73 (Dual vector matroid via orthogonal complement)**.** Let A and A' be matrices over a field F with the same column index set E, and suppose

$$\mathrm{row}(A') = \mathrm{row}(A)^\perp \subseteq F^E.$$

Let $M(A)$ and $M(A')$ be the vector matroids represented by A and A'. Then

$$M(A') = M(A)^*.$$

Proof. Let $F \subseteq E$.

(⇒) Suppose F is dependent in $M(A)$. Then there exists a nonzero vector $c \in F^F$ such that $A_F c = 0$. Extend c by zero outside F (still denoted c). The condition $Ac = 0$ means each row r of A satisfies $r \cdot c = 0$, hence $c \in \mathrm{row}(A)^\perp = \mathrm{row}(A')$. Write

$$c = \sum_i \lambda_i r'_i,$$

where the r'_i are rows of A' and not all λ_i are zero. For every $e \in E \setminus F$ we have $c_e = 0$, so

$$\left(\sum_i \lambda_i r'_i\right)\Big|_{E\setminus F} = 0.$$

Hence the rows of A' indexed by $E \setminus F$ admit a nontrivial linear combination giving the zero row, so $E \setminus F$ is dependent in $M(A')$.

(⇐) The same argument with A and A' interchanged, using $\mathrm{row}(A) = (\mathrm{row}(A')^\perp)$, shows that if $E \setminus F$ is dependent in $M(A')$, then F is dependent in $M(A)$.

Thus

$$F \text{ dependent in } M(A) \iff E \setminus F \text{ dependent in } M(A'),$$

which is the defining property of duality. □

Theorem 74 (Dual of standard representation corresponds to dual matroid)**.** *Let M be a binary matroid on ground set $E = X \cup Y$, with standard representation B so that*

$$A = [\,\mathbf{1}_X \mid B\,].$$

Let $B^* := -B^\intercal$ and

$$A^* := [\,\mathbf{1}_Y \mid B^*\,].$$

Then $M(A^*) = M(A)^* = M^*$.

Proof. By Lemma 70 and Lemma 71, if $U = \mathrm{row}(A)$ then $U^\perp$ has the form

$$U^\perp = \{\,(bB^*, b) \mid b \in \mathbb{F}_2^Y\,\}.$$

By Lemma 72, we have

$$\mathrm{row}(A^*) = U^\perp = \mathrm{row}(A)^\perp.$$

Therefore, by Lemma 73, the column-matroid $M(A^*)$ is the dual of $M(A)$:

$$M(A^*) = M(A)^* = M^*.$$

□

Lemma 75. The dual matroid of a regular matroid is also a regular matroid.

Proof. Let M be a regular matroid. We wish to show that M^* is also regular.

Take a standard $\mathbb{Z}_2$-representation matrix B of M. By Lemma 33, since M is regular, there exists a TU signing B' of B: B' is a matrix over $\mathbb{Q}$ that is TU, and $|B'(i,j)| = B(i,j)$ for all entries. So M is represented (over $\mathbb{Q}$) by a TU matrix B' whose pattern of zero and non-zero entries is exactly that of B.

From Theorem 74, if a matroid M has standard representation matrix B, then its dual M^* has the standard representation matrix $B^* = -B^\intercal$. The TU signing of this dual standard matrix, $(B')^* = -(B')^\intercal$, preserves total unimodularity, so $(B')^*$ is a TU matrix whose support is exactly B^*.

Since we have just exhibited a TU signing of M^* (i.e., $(B')^*$), the dual matroid M^* is regular by Lemma 33. □

Theorem 76. *Every cographic matroid is regular.*

Proof. We know that all graphic matroids are regular by Theorem 69. Recall that we say a matroid is cographic if its dual is graphic. So it suffices to show regularity is preserved under duals, which we showed in Lemma 75. □

7. Conclusion

Definition 77. Any graphic matroid is good. Any cographic matroid is good. Any matroid isomorphic to R10 is good. Any 1-sum (in the sense of Definition 35) of good matroids is a good matroid. Any 2-sum (in the sense of Definition 40) of good matroids is a good matroid. Any 3-sum (in the sense of Definition 46) of good matroids is a good matroid.

Corollary 78. Any good matroid is regular. This is a corollary of the easy direction of the Seymour theorem.

Proof. Structural induction using theorems 69, 76, 68, 38, 44, and 62. □

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