1. Equation 1518

Our carrier set will be the (nonabelian) free group G on countably many letters $x_1, x_2, \ldots$. The functional equation for 1518 is

$$f(f(f(h)h^{-1})hf(1)^{-1}) = hf(1)^{-1}.$$

Think of this as saying that once $(1, x_1), (a, b), (ba^{-1}, c) \in f$, then $(cax_1^{-1}, ax_1^{-1}) \in f$. A special family built from this rule is

$$E_0 = \{(1, x_1), (x_1, x_2), (x_2x_1^{-1}, x_1^{-1}), (x_1^{-1}, 1), (x_2x_1^{-2}, x_1^{-2}), (x_1^2, x_3), (x_3x_1^{-2}, x_1^{-1})\}.$$

Let $\mathscr{E}$ be the collection of subsets $E \subseteq G^2$ subject to the following conditions:

(1) E is finite.
(2) E is a function.
(3) $E_0 \subseteq E$.
(4) If $(a, b), (ba^{-1}, c) \in E$, then $(cax_1^{-1}, ax_1^{-1}) \in E$.
(5) If $(a, b) \in E$ and $b = x_1$, then $ba^{-1} \in \mathrm{dom}(E)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a, b) \in E$ for some $b \in G$.

If $ba^{-1} \in \mathrm{dom}(E)$, then we are done by property (4). So we may reduce to the case when $ba^{-1} \notin \mathrm{dom}(E)$.

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be the set of pairs in E such that $b_ia_i^{-1} = ba^{-1}$. Since $ba^{-1} \notin \mathrm{dom}(E)$, we know that $a_i \neq x_1^2$ from our choice of E_0. Also, we know that $b_i \neq x_1$ by condition (5).

Let c be a generator of G that does not appear in the normal form of any coordinate in E and take

$$E' = E \cup \{(ba^{-1}, c)\} \cup \{(ca_ix_1^{-1}, a_ix_1^{-1})\}_{i \in [1,n]}.$$

Conditions (1), (2), and (3) are clear for E'. Condition (5) for E' follows from condition (5) for E and the fact that $a_ix_1^{-1} \neq x_1$ since $a_i \neq x_1^2$. Condition (4) is a quick case analysis, mostly using the newness of the generator c; but note that if $c(ba^{-1})^{-1} = ca_ix_1^{-1}$, then $b_i = x_1$, which we already stated is impossible.

Case 2: Assume $a \notin \mathrm{dom}(E)$. If $(x, y) \in E$ with $yx^{-1} = a$, then applying Case 1 to x, we get $a \in \mathrm{dom}(E)$, and then reduce to Case 1. Thus, we may assume there is not such pair. Then letting b be any generator of G not in the normal form of any element in E or of a, take $E' := E \cup \{(a, b)\}$, and again reduce to Case 1. $\square$

The functional equation for 47 is

$$f^3(1) = 1.$$

Taking $E_0 \cup \{x_2, x_4)\}$ as the initial seed works.

Equation 614's functional equation is

$$f^2(f(x_1^{-1})x_1) = 1.$$

Again taking $E_0 \cup \{(x_2, x_4)\}$ works.

The functional equation for 817 is

$$f(x_1^2) = 1.$$

Using $E_0 \cup \{(x_3, x_4)\}$ works.

The functional equation for 3862 is

$$x_1 = f(f(x_1)^{-1})f(x_1).$$

Taking $E_0 \cup \{(x_2^{-1}, x_4), (x_4 x_2 x_1^{-2}, x_2 x_1^{-2}), (x_4 x_2 x_1^{-3}, x_2 x_1^{-3})\}$ works.