Local cohomology, Macaulay2, and formalization in commutative algebra

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Local cohomology

Cohomology theory introduced by Grothendieck in the 1960s

"Algebraic child of geometric parents"

Motivation

Algebraic variety

Given polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$,

$$\mathbb{V}(f_1, \ldots, f_n) = \{a \in \mathbb{C}^n \mid f_1(a) = f_2(a) = \cdots = f_m(a) = 0\}$$

Motivation

Example

$$\mathbb{V}(x^2, xy) = \mathbb{V}(x^2) = \mathbb{V}(x)$$

Motivation

Question

Can $V = \mathbb{V}(xz, yw, xw, yz) \subseteq \mathbb{C}^4$ be defined by three equations?

Observe

$$(0, 1, 1, 0) \in \mathbb{V}(xz, yw, xw) \setminus V$$

Motivation

Question

Can $V = \mathbb{V}(xz, yw, xw, yz) \subseteq \mathbb{C}^4$ be defined by three equations?

Observe

$$(\underline{xz + yw})xz - (\underline{xw})(yz) = (xz)^2$$

Hence if $xz + yw = xw = 0$, then $xz = 0 \implies yw = 0$

$$\implies V = \mathbb{V}(xz + yw, xw, yz)$$

Motivation

$$V = \mathbb{V}(xz + yw, xw, yz)$$

Question

What about two equations?

Local cohomology

Given polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n] =: R$,

the local cohomology modules with support in $I = \langle f_1, \ldots, f_m \rangle$ are R-modules

$$H_I^0(R),\ H_I^1(R), \ldots,\ H_I^i(R),\ \ldots$$

Local cohomology

Given

- A Noetherian ring R,
- An ideal I of R, and
- An R-module M,

the local cohomology modules of M with support in I are R-modules

$$H_I^0(M),\ H_I^1(M), \ldots,\ H_I^i(M),\ \ldots$$

Local cohomology

Theorem

$H^m_I(R) \neq 0 \implies \mathbb{V}(I)$ **cannot** be defined by fewer than m equations

Proof IOU

Fact

$$H^3_{\langle xz, yw, xw, yz\rangle}(\mathbb{C}[x,y,z,w]) \neq 0$$

Proof IOU

Local cohomology

Question

Why "local"?

Rough motivation

Relative singular cohomology $H^\bullet(\mathcal{X}, \mathcal{X} \setminus \mathcal{Z}, \mathbb{Z})$ can be realized as local cohomology

Fact

$$H^3_{\langle xz, yw, xw, yz \rangle}(\mathbb{C}[x, y, z, w]) \neq 0$$

Proof IOU

Second vanishing theorem

$$H_I^{\dim R - 1}(R) \neq 0 \text{ if and only if } \mathbb{V}(I) \setminus \underline{0} \text{ connected}$$

Macaulay2 computer algebra system

Macaulay2

- Created by Dan Grayson and Mike Stillman
- Open source computer algebra system
- Commutative algebra and algebraic geometry
- Research tool
- Modern results implemented
- Frequent development workshops
- Packages published with documentation
- Journal of Software for Algebra and Geometry

Definitions and steps to formalization

Definitions and steps to formalization

- Noetherian ring R
- $I \subseteq R$ ideal
- R-module M

$$I \supseteq I^2 \supseteq I^3 \supseteq \cdots \quad \text{induces} \quad R/I \twoheadleftarrow R/I^2 \twoheadleftarrow R/I^3 \twoheadleftarrow \cdots$$

which induces direct limit system

$$\cdots \to \operatorname{Ext}^i_R(R/I^t, M) \to \operatorname{Ext}^i_R(R/I^{t+1}, M) \to \cdots$$

$$H^i_I(M) := \varinjlim \operatorname{Ext}^i_R(R/I^t, M)$$

Definition 1

$$H_I^i(M) := \varinjlim \mathrm{Ext}_R^i(R/I^t, M)$$

- Projective(/injective) resolutions
- Hom functor
- Ext functor
- Direct limits

Definitions and steps to formalization

Given $f \in R$,

$$K^\bullet(f; R) : \quad 0 \to R \xrightarrow{\cdot f} R \to 0$$

- Natural maps $K^\bullet(f^t; M) \to K^\bullet(f^{t+1}; M)$
- $K^\bullet(f; M) := K^\bullet(f; R) \otimes_R M$
- $K^\bullet(f_1, \ldots, f_m; M) := K^\bullet(f_1; M) \otimes_R \cdots \otimes_r K^\bullet(f_m; M)$

Definition 2

If $I = \langle f_1, \ldots, f_m \rangle$, let $\underline{f}^t = f_1^t, \ldots, f_m^t$

$$H^i_I(R) := H^i(\varinjlim K^\bullet(\underline{f}^t; M))$$

- Tensor products of complexes

Definitions and steps to formalization

Definition 2

If $I = \langle f_1, \ldots, f_m \rangle$, let $\underline{f}^t = f_1^t, \ldots, f_m^t$

$$H_I^i(R) := H^i(\varinjlim K^\bullet(\underline{f}^t; M))$$

$$= \varinjlim H^i(K^\bullet(\underline{f}^t; M))$$

Definition 3

$H_I^i(M)$ is i-th cohomology of

$$0 \to M \to \oplus M_{f_i} \to \oplus M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0$$

- Cohomology commutes with direct limits
- Direct limit of $M \xrightarrow{\cdot f} M \xrightarrow{\cdot f} M \to \cdots$ is M_f

Definition 3

$H^i_I(M)$ is i-th cohomology of

$$0 \to M \to \bigoplus M_{f_i} \to \bigoplus M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0$$

$\implies$ IOU

Definitions and steps to formalization

$$\Gamma_I(M) := \{x \in M \mid I^N \cdot x = 0 \text{ for some } N > 0\}$$

Definition 0

$H^i_I(M)$ is the right derived functor of Γ_I

- Injective resolutions