Topological properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ and Euclidean models

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1 Topology of $\mathbb{Q}_p$ vs Topology of $\mathbb{R}$

Both $\mathbb{R}$ and $\mathbb{Q}_p$ are normed fields and complete metric spaces, both are completions of $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in both of them, they are seperable.
An open ball in $\mathbb{Q}_p$ with center a and radius r is denoted by

$$B_{(a,r)} := \{x \in \mathbb{Q}_p \mid \|a - x\|_p < r\}$$

since

$$\{\|x - y\|_p \mid x, y \in \mathbb{Q}_p\} = \{p^n \mid n \in \mathbb{Z}\} \cup \{0\}$$

we only need to consider the balls of radi $r = p^n$, where $n \in \mathbb{Z}$.
The sphere with center a and radius r is denoted by

$$S_{(a,r)} = \{x \in \mathbb{Q}_p \mid \|x - a\|_p = r\}$$

Proposition 1.1. *The Sphere $S_{(a,r)}$ is an open set in $\mathbb{Q}_p$.*

Proof. Let $x \in S_{(a,r)}$, choose $\varepsilon < r$. We now show that $B_{(x,\varepsilon)} \subset S_{(a,r)}$.
Let $y \in B_{(x,\varepsilon)} \Rightarrow \|x - y\|_p < \|x - a\|_p = r$ and by Proposition 1.15 (Katok) follows $\|y - a\|_p = \|x - a\|_p = r$ which means that $y \in S_{(a,r)}$.
Since x and y were arbitrary it follows that

$$S_{(a,r)} = \bigcup_{x \in S_{(a,r)}} B_{(x,\varepsilon)}$$

Therefore the sphere $S_{(a,r)}$ is a union of open sets, so it is open itself. $\square$

Proposition 1.2. *Open balls in $\mathbb{Q}_p$ are open and closed.*

Proof. $B_{(a,r)}$ is closed $\Leftrightarrow$ ${B_{(a,r)}}^c = \{x \in \mathbb{Q}_p \mid \|x-a\|_p \geq r\}$ is open. We know

$$ {B_{(a,r)}}^c = S_{(a,r)} \cup D $$

where $D := \{x \in \mathbb{Q}_p \mid \|x-a\|_p > r\}$
Because of the last Proposition it sufficies to show that D is open.
So, let $y \in D, \|y-a\|_p =: r_1 > r$ We claim: $B_{(y,r_1-r)} \subset D$.
Otherwise there exists an $x \in B_{(y,r_1-r)}$ such that $\|x-a\|_p \leq r$ but using the triangle inequality, leads to

$$ r_1 = \|y-a\|_p = \|y-x+x-a\|_p \leq \|y-x\|_p + \|x-a\|_p < r + (r_1 - r) = r $$

which is a contradiction to our assumption $r_1 < r$.
Therefore, $B_{(y,r_1-r)} \subset D$, so $D = \bigcup_{y\in D} B_{(y,r_1-r)}$ which means that D is open. □

Hence, since $B_{(a,r)}$ is closed, the open balls in $\mathbb{Q}_p$ have no boundary and in particular $S_{(a,r)}$ is not the boundary of $B_{(a,r)}$. In addition we get that

$$ \overline{B}_{(a,p^n)} \neq \overline{B_{(a,p^n)}} = B_{(a,p^n)} $$

in fact we have

$$ \overline{B}_{(a,p^n)} = \{x \in \mathbb{Q}_p \mid \|x-a\|_p \leq p^n\} = \{x \in \mathbb{Q}_p \mid \|x-a\|_p < p^{n+1}\} = B_{(a,p^{n+1})} $$

Proposition 1.3. *Every point of a ball is its center, i.e.*

$$ \forall b \in B_{(a,r)} \text{ we have } B_{(a,r)} = B_{(b,r)} $$

Proposition 1.4. *Two balls in $\mathbb{Q}_p$ have a non empty intersection if and only if one is contained in the other, i.e.*

$$ B_{(a,r)} \cap B_{(b,s)} \neq \emptyset \Leftrightarrow B_{(a,r)} \subset B_{(b,s)} \text{ or } B_{(b,s)} \subset B_{(a,r)} $$

Proof. $\Leftarrow$ is clear
$\Rightarrow$ let $y \in B_{(a,r)} \cap B_{(b,s)}$ w.l.o.g assume $r \leq s$, by the Proposition before we have

$$ B_{(a,r)} = B_{(y,r)} \subset B_{(y,s)} = B_{(b,s)} $$

□

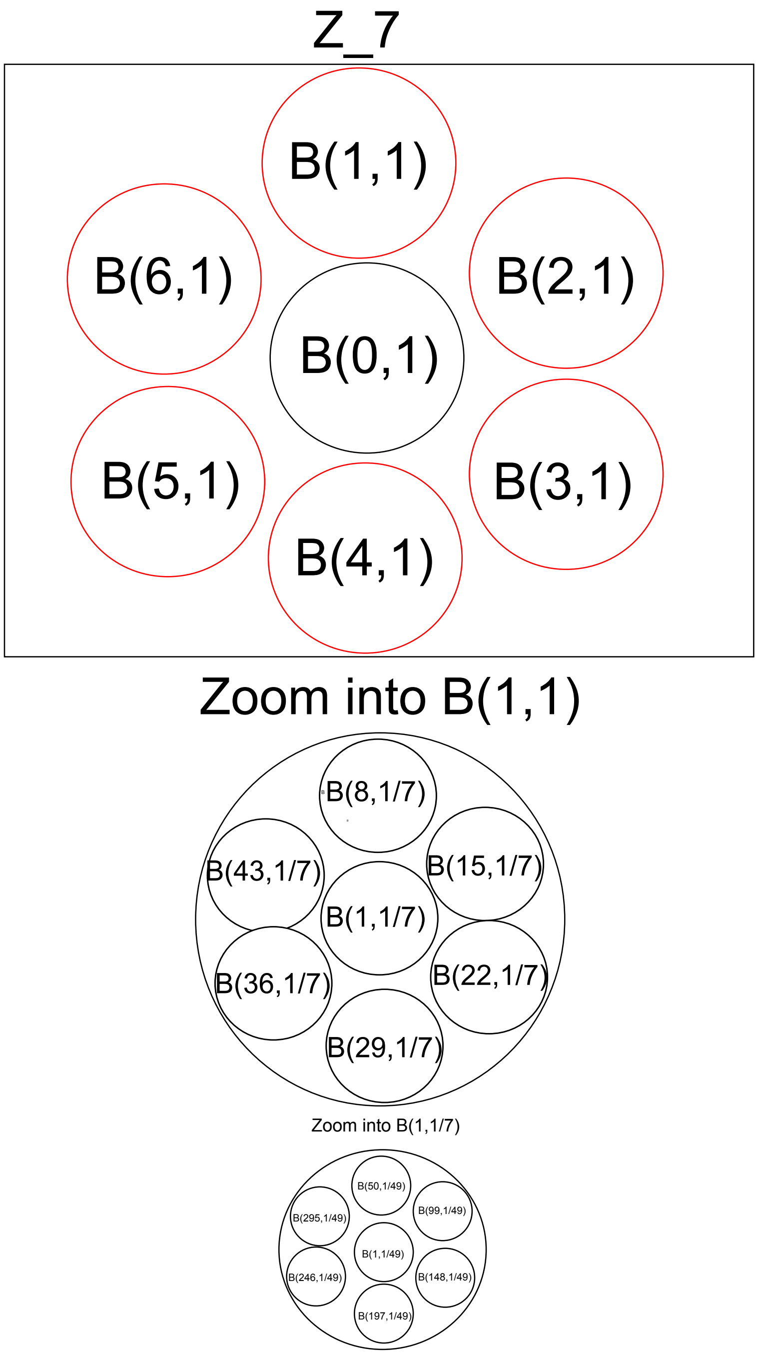
Z_7
B(1,1)
B(6,1)
B(2,1)
B(0,1)
B(5,1)
B(3,1)
B(4,1)
Zoom into B(1,1)
B(8,1/7)
B(43,1/7)
B(15,1/7)
B(1,1/7)
B(36,1/7)
B(22,1/7)
B(29,1/7)
Zoom into B(1,1/7)
B(50,1/49)
B(295,1/49)
B(99,1/49)
B(1,1/49)
B(246,1/49)
B(148,1/49)
B(197,1/49)

Figure 1: Z_7

Taking Z_7 as example we see that $S_{(0,1)} = \bigcup_{x\in\{1,\dots,6\}} B_{(x,1)}$
in addition it holds that

$$\forall x, y \in Z_7 \text{ s.t. } x \in y + 7^k Z_7 \text{ we have } B(x, 7^{-(k-1)}) = B(y, 7^{-(k-1)})$$

let $x \in Z_7$ be arbitrary, $k \in \mathbb{N}$ then we can observe that

$$B_{(x,7^{-k})} = \bigcup_{j\in\{0,\dots 6\}} B(x + j * 7^{k+1}, 7^{-(k+1)})$$

We can generalize this :
Let p be a prime number, then we have that $S_{(0,1)} = \bigcup_{x\in\{1,\dots,p-1\}} B_{(x,1)}$ in addition we have that

$$\forall x, y \in Z_p \text{ s.t. } x \in y + 7^k Z_p \text{ we have } B(x, p^{-(k-1)}) = B(y, p^{-(k-1)})$$

and for an arbitrary element $x \in Z_p$, for all $k \in \mathbb{N}$ it holds that

$$B_{(x,p^{-k})} = \bigcup_{j\in\{0,\dots p-1\}} B(x + j * p^{k+1}, p^{-(k+1)})$$

Proposition 1.5. *The sphere $S_{(a,r)}$ is open and closed*

Proof. We have already shown that every sphere is open. We observe that $S_{(a,r)} = \overline{B}_{(a,r)} \cap {B_{(a,r)}}^c$ which is closed because it is a finite intersection of closed subsets. □

Proposition 1.6. *The set of all balls in $\mathbb{Q}_p$ is countable.*

Proof. Let $B_{(a,r)}$ be an arbitrary ball in $\mathbb{Q}_p$. We know that $r = p^{-s}$ for some $s \in \mathbb{Z}$. Since $a \in \mathbb{Q}_p$ there exist $m \in (\mathbb{Z})$ s.t. $a_m \neq 0$ and $a = \sum_{n=m}^{\infty} a_n p^n$. Let $a_0 := \sum_{n=m}^{s} a_n p^n$, obviously $a_0 \in \mathbb{Q}$ and we have $\|a - a_0\|_p < p^{-s}$ which means that $a_0 \in B_{(a,p^{-s})}$. As before, we have $B_{(a,r)} = B_{(a_0,p^{-s})}$.
Therefore both, the set of radii and the set of centers of balls in $\mathbb{Q}_p$, are countable which leads to the fact that the set of balls in $\mathbb{Q}_p$ is countable. □

Theorem 1.7. *The set $\mathbb{Z}_p$ is compact and the space $\mathbb{Q}_p$ is locally compact.*

Proof. We know that $\mathbb{Z}_p$ is sequentially compact, since it is a metric space it is therefore compact. Because $\mathbb{Z}_p = \overline{B}_{(0,1)} = B_{(0,p)}$ it follows that every ball in $\mathbb{Q}_p$ is compact. So $\mathbb{Q}_p$ is a locally compact space. □

Theorem 1.8. $\mathbb{N}$ *is dense in* $\mathbb{Z}_p$.

Proof. let $x = \ldots a_2 a_1 a_0 \in \mathbb{Z}_p$ for all $n \in \mathbb{N}$ define

$$x_n := \ldots 00 a_n a_{n-1} \ldots a_0 = \sum_{i=0}^{n} a_i p^i \in \mathbb{N}$$

we obtain $\|x - x_n\|_p < p^{-n}$ □

Theorem 1.9. *The space* $\mathbb{Q}_p$ *is totally disconnected.*

Proof. We show that for an arbitrary element $a \in \mathbb{Q}_p$ the connected component C_a of a is equal to $\{a\}$. Let a be arbitrary and suppose $C_a \supsetneq \{a\}$ therefore there exist $n \in \mathbb{N}$ such that $B_{(a,p^{-n})} \cap C_a \neq C_a$. But then we have

$$C_a = (B_{(a,p^{-n})} \cap C_a) \cup ((\mathbb{Q}_p \setminus B_{(a,p^{-n})}) \cap C_a)$$

which is the disjoint union of two open subsets. Therefore C_a is not connected, which is a contradiction.

□

2 Cantor Set Models of $\mathbb{Z}_p$

We start with a repetition of the classical Cantor set $C \subset [0,1]$ and explain its basic properties. We will observe that $\mathbb{Z}_p$ is homeomorphic to C for every prime number p. In the case $p = 2$ there is a natural homeomorphism $\mathbb{Z}_2 \cong C$ using the triadic expansion of real numbers. In the case $p > 2$ there is a natural homeomorphism $C^{(p)} \cong \mathbb{Z}_p$, where $C^{(p)} \subset [0,1]$ is obtained by a similar construction as the one of the classical Cantor set C.

This allows us to reduce the initial claim that every $\mathbb{Z}_p$ is homeomorphic to C to the statement that $C^{(p)}$ and C are homeomorphic.

Definition 2.1 (Cantor Set)**.** Let $A = \bigcup_{k \in \mathbb{Z}} [2k, 2k+1]$ and $C_0 := [0,1]$. We define by induction

$$C_n := C_{n-1} \cap \left(3^{-n} A\right).$$

Then the **Cantor set C** is given by

$$C := \bigcap_{n \geq 0} C_n$$

Each set C_n consists of 2^n closed intervals of length 3^{-n} and C_{n+1} is obtained from C_n by removing the middle third in of each of these intervals.

Lemma 2.1 (Properties of the Cantor Set C)**.** *The Cantor set $C \subset [0,1]$ satisfies the following:*

1. *C is compact.*
2. *The Cantor set has vanishing Lebesgue measure, i.e. $|C| = 0$ where $|\cdot|$ denotes the Lebesgue measure.*
3. *The Cantor set is perfect (see definition below).*
4. *The Cantor set is uncountable.*

Definition 2.2 (Perfect Set)**.** A closed set E is called **perfect**, if for every $x \in E$ there exists a sequence $(x_n) \subset E - \{x\}$ converging to x.

Proof. C is clearly bounded and closed, since each C_n is closed, and therefore C is compact.

Since C_n is the union of 2^n intervals of length 3^{-n} we have $|C_n| = (2/3)^n$ and since $C_n \supset C_{n+1}$ for all n we conclude

$$|C| = \lim_{n\to\infty} |C_n| = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$$

For the third statement note that $\partial C_n \subset C$. Now pick $x \in C$ arbitrary. We can choose $x_n \in \partial C_n - \{x\}$ such that $|x_n - x| \leq 3^{-n}$ and therefore $x_n \to x$.

We show that the third statement implies the last one. So let $E \subset \mathbb{R}$ be any nonempty perfect subset and assume $E = \{e_i\}_{i=1}^{\infty}$ is countable. Define $E_n := E - \{e_n\}$. Choose $x_1 \in E_1$ and let I_1 be a finite open interval such that $x_1 \in I_1$ and $e_1 \notin \overline{I}_1$. Since E is perfect we have $I_1 \cap E_2 \neq \emptyset$. Then pick $x_2 \in I_2 \cap E_2$ and let I_2 be an open interval such that $x_2 \in I_2 \subset I_1$ and $e_2 \notin \overline{I}_2$. Continuing in this way we obtain a decreasing sequence (I_n) of intervals such that $e_n \notin \overline{I}_n$. On the other hand we observe

$$\bigcap_{n\geq 1} E \cap \overline{I}_n \neq \emptyset$$

since all sets $E \cap \overline{I}_n$ are compact and nonempty. This contradicts our assumption that E is countable.

□

Definition 2.3 (d-Adic Expansion)**.** Let $d \in \mathbb{Z}^+$ and $x \in [0,1]$. A d**-adic expansion** of x is given by

$$x = \sum_{n=1}^{\infty} x_n d^{-n}$$

where $x_n \in \{0, 1, \dots, d-1\}$. It is easy to check that every $x \in [0,1]$ admits a unique d-adic expansion, when we enforce the digit sequence $\{x_k\}$ to have not finite support. We denote the d-adic expansion by $[x]_d = 0, x_1 x_2 \dots$.

For $d = 10$ this leads to the usual decimal expansion of real numbers and we have for example $0.1 = 0.0999999\dots$. By definition $[1/10]_{10}$ is the second variant.

Lemma 2.2 (Characterization of C via triadic expansion)**.** *Let $x \in [0,1]$ and $[x]_3 = 0, x_1 x_2 \dots$ be the triadic expansion of x. Then it holds $x \in C$ if and only if $x_n \in \{0, 2\}$ for every n.*

Proof. By induction on n, it follows immediately from the definitions that $x \in C_n$ if and only if $x_1, \dots, x_n \in \{0, 2\}$. □

Theorem 2.3 (Homeomorphism of $\mathbb{Z}_2$ and C)**.** *$\mathbb{Z}_2$ equipped with the 2-adic norm $|\cdot|_2$ is homeomorphic to C equipped with the absolute value as norm. An explicit homeomorphism is given by*

$$\Phi : \mathbb{Z}_2 \longrightarrow C, \qquad \sum_{n=0}^{\infty} a_n 2^n \mapsto \sum_{n=0}^{\infty} (2a_n) 3^{-(n+1)}.$$

Proof. Clearly, Φ is bijective. Let $x = \sum x_n 2^n$, $y = \sum y_n 2^n \in \mathbb{Z}_2$. Then it holds

$$\begin{aligned} |x - y|_2 \leq 2^{-k} &\Leftrightarrow x_n = y_n \text{ for } n \leq k \\ &\Leftrightarrow \text{The first } k \text{ digits in } [\Phi(x)]_3 \text{ and } [\Phi(y)]_3 \text{ are equal} \\ &\Leftrightarrow |\Phi(x) - \Phi(y)| \leq 3^{-k} \end{aligned}$$

This shows that Φ is a homeomorphism. □

Now we will extend our discussion to the case of $\mathbb{Z}_p$ for a prime number $p > 2$. The results will follow completely analog to the case $p = 2$ and therefore our presentation will be briefer.

Definition 2.4 (Cantor Set $C^{(p)}$)**.** We define a variant of the classical Cantor set. Let p be a prime number, $A = \bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$ and $C_0^{(p)} := [0,1]$. We define inductively

$$C_n^{(p)} := C_{n-1}^{(p)} \cap \big((2p-1)^{-n} A\big)$$

and

$$C^{(p)} := \bigcap_{n\geq 0} C_n^{(p)}.$$

The set $C_n^{(p)}$ consists of $[(2p-1)/2]^n$ disjoint open intervals of length $(2p-1)^{-n}$ and $C_{n+1}^{(p)}$ is obtained by subdividing each of these intervals into $2p-1$ equal subintervals and then deleting every second open interval.

Lemma 2.4 (Properties of $C^{(p)}$)**.** *The Cantor set $C^{(p)}$ satisfies the following properties:*

1. *$C^{(p)}$ is a compact and perfect subset of the real line with vanishing Lebesgue measure. In particular $C^{(p)}$ is uncountable.*

2. *$C^{(p)}$ contains all $x \in [0,1]$ whose $(2p-1)$-adic expansion $[x]_{2p-1} = 0, x_1x_2x_3\ldots$ contains only even digits $\{x_n\}$.*

Proof. The proof is completely analogue to the case $p = 2$. □

Theorem 2.5 (Homeomorphism of $\mathbb{Z}_p$ and $C^{(p)}$)**.** *$\mathbb{Z}_p$ equipped with the p-adic norm $|\cdot|_p$ is homeomorphic to $C^{(p)}$ equipped with the absolute value as norm. An explicit homeomorphism is given by*

$$\Phi_p : \mathbb{Z}_p \longrightarrow C, \qquad \sum_{n=0}^{\infty} a_n p^n \mapsto \sum_{n=0}^{\infty} (2a_n)(2p-1)^{-(n+1)}.$$

Proof. The proof is the same as for $p = 2$. It is clear that Φ_p is bijective. Let $x = \sum x_n p^n, y = \sum y_n p^n \in \mathbb{Z}_p$. Then it holds

$$\begin{aligned} |x-y|_2 \leq p^{-k} &\Leftrightarrow x_n = y_n \text{ for } n \leq k \\ &\Leftrightarrow \text{The first } k \text{ digits in } [\Phi(x)]_{2p-1} \text{ and } [\Phi(y)]_{2p-1} \text{ are equal} \\ &\Leftrightarrow |\Phi(x) - \Phi(y)| \leq (2p-1)^{-k} \end{aligned}$$

This shows that Φ_p is a homeomorphism.

□

Theorem 2.6 (Topological equivalence of the spaces $\mathbb{Z}_p$)**.** *The spaces* $\mathbb{Z}_2$ *and* $\mathbb{Z}_p$ *are homeomorphic*

Recall that all spaces $\mathbb{Z}_p$ are totally disconnected. Consequently, the classical Cantor set C and its variants $C^{(p)}$ are totally disconnected and therefore the theorem follows from the following Lemma.

Lemma 2.7. *Any compact perfect totally disconnected subset* E *of the real line is homeomorphic to the Cantor set* C

Proof. Let $m := \inf E$ and $M := \sup E$. We are going to define a function $F : [m, M] \to [0, 1]$ such that F maps E homeomorphic on C. We will construct this function on the complements $[m, M]\backslash E \to [0, 1]\backslash C$ first, since the complements are both countable unions of open intervals and therefore much less complicated then the sets E and C themselves. Since both complements are dense we can extend this map by continuity to a map $F : [m, M] \to [0, 1]$.

The connected subsets of the real line are precisely the intervals. Therefore $[m, M]\backslash E$ is the disjoint union of infinite but countably many open intervals (they are the connected components of $[m, M]\backslash E$) and the same is true for $[0, 1]\backslash C$. Let $\mathcal{I}$ be the collection of the intervals whose union is $[m, M]\backslash E$ and $\mathcal{J}$ be the collection whose union is $[0, 1]\backslash C$. We start with the construction of an appropriate bijection

$$\Theta : \mathcal{I} \to \mathcal{J}.$$

Let $I_1 \in \mathcal{I}$ be an interval of maximal length and define $\Theta(I_1) = (1/3, 2/3)$. Next choose intervals I_{21} and I_{22} to the left and right of I_1 such that they have maximal length and define $\Theta(I_{21}) = (1/9, 2/9)$ and $\Theta(I_{22}) = (7/9, 8/9)$. Continuing this process defines Θ on the whole set $\mathcal{I}$, since $\mathcal{I}$ contains only finitely many sets of length greater then some fixed $\epsilon > 0$ and since any two intervals in $\mathcal{I}$ or in $\mathcal{J}$ have different endpoints (as E and C are perfect). It is clear from the construction that Θ is bijective and order preserving in the sense that if I is to the left of I', then $\Theta(I)$ is to the left of $\Theta(I')$.

Now define F as follows: For $I \in \mathcal{I}$ let $F|_I : I \to \Theta(I)$ be the unique linear increasing map, which maps I bijectively onto $\Theta(I)$. Since E and C are totally disconnected, they are nowhere dense and thus there exists at most one continuation $F : [m, M] \to [0, 1]$. Now it follows from our construction of Θ that a well-defined continuation $F : [m, M] \to [0, 1]$ is given by

$$F(x) = \sup\{\ F(y)\ :\ y \notin E,\, y \leq x\ \}$$

Let $f := F|_E$. Then $f : E \to C$ is a monotone increasing, continuous bijection and we need to show that $g := f^{-1}$ is continuous. Note that g is again monotone increasing. Let $x \in C$ and $x_n \to x$ converge to x. This sequence contains a monotone subsequence and thus we may assume wlog that the sequence x_n itself is monotone increasing. Clearly,

$$y := \lim_{n\to\infty} g(x_n) = \sup_{n\geq 1} g(x_n) \leq g(x)$$

Now assume $y < g(x)$. Since E is closed, we have $y \in E$ and $g^{-1}(y) < x$. This implies that $y < x_n$ for large n and by monotonicity $y < g(x_n)$. This contradicts the definition of y and shows continuity of g.

□

3 Euclidean models of $\mathbb{Z}_p$

3.1 Linear models of $\mathbb{Z}_p$

Let us choose $b \in (1, \infty)$ and and use it as numeration base in $[0, 1]$, i.e. we write

$$[0,1] \ni a = \frac{a_0}{b} + \frac{a_1}{b^2} + \ldots = \sum_{i=0}^{\infty} \frac{a_i}{b^{i+1}}$$

with $0 \leq a_i < b$, $a_i \in \mathbb{N}$.
We consider

$$\psi := \psi_{b,p} : \mathbb{Z}_p \to [0,1] \qquad \sum_{i=0}^{\infty} a_i p^i \mapsto \alpha \sum_{i=0}^{\infty} \frac{a_i}{b^{i+1}} \tag{1}$$

with $\alpha \in \mathbb{R}^+$ chosen such that $\max_{\zeta\in\mathbb{Z}_p} \psi(\zeta) = 1$.

Proposition 3.1. *Let $\psi : \mathbb{Z}_p \to [0,1]$ be defined as in (1) with $b > 1$ and let p be a prime number. Then*

1.

$$\alpha = \frac{b-1}{p-1}$$

2. ψ and its inverse are always continuous;

3. *ψ is injective if $b > p$ (and in this case it is a homeomorphism onto its image $\psi(\mathbb{Z}_p)$).*

Proof. 1. Since the a_i's are all positive numbers, $\max_{\zeta\in\mathbb{Z}_p}\psi(\zeta)$ is attained when these are maximal (thus $a_i = p-1$ for all $i \in \mathbb{N}$). So take $\zeta = \sum_{i=0}^{\infty}(p-1)p^i = -1$. Then

$$1 = \alpha(p-1)\sum_{i=0}^{\infty}\frac{1}{b^{i+1}} = \alpha(p-1)\left(\frac{1}{1-\frac{1}{b}} - 1\right) = \alpha\frac{p-1}{b-1}$$

and thus the claim follows.

2. This is analogous to the proof of Theorem 2.6. Let $x = \sum_{n=0}^{\infty} x_n p^i$ and $y = \sum_{n=0}^{\infty} y_n p^i \in \mathbb{Z}_p$. Then

$$\begin{aligned}|x-y|_p \le p^{-k} &\Leftrightarrow x_n = y_n \text{ for } n \le k\\ &\Leftrightarrow \text{ The first } k \text{ digits in } [\psi(x)]_b \text{ and } [\psi(y)]_b \text{ are equal}\\ &\Leftrightarrow |\psi(x)-\psi(y)| \le b^{-k}\end{aligned}$$

and since both b, $p \ge 1$ we have always continuity for both maps.

3. We will have injectivity if $\sum_{i>k}(p-1)p^i$, $p^k \in \mathbb{Z}_p$ have distinct images in $[0,1]$ with $\psi(\sum_{i>k}(p-1)p^i) < \psi(p^k)$.

$$\begin{aligned}\psi(\sum_{i>k}(p-1)p^i) &= \frac{b-1}{p-1}\sum_{i=k+1}^{\infty}\frac{p-1}{b^{i+1}} = (b-1)(\sum_{i=0}^{\infty}\frac{1}{b^i} - \sum_{i=0}^{k+2}\frac{1}{b^i}) =\\ &= (b-1)\left(\frac{1}{1-\frac{1}{b}} - \frac{1-\frac{1}{b^{k+2}}}{1-\frac{1}{b}}\right) =\\ &= (b-1)\frac{b}{b-1}(1-\frac{b^{k+2}-1}{b^{k+2}}) = \frac{b}{b^{k+2}} = b^{-k-1}\\ \psi(p^k) &= \frac{b-1}{p-1}\frac{1}{b^{k+1}} = \frac{b-1}{p-1}b^{-k-1}\end{aligned}$$

Thus we must have $\frac{b-1}{p-1} > 1$, i.e. $b > p$.

□

Remark. As seen in the previous section, the Cantor set C^p corresponds to $\psi_{2p-1,p}$.

When $b > p$, ψ is thus an homeomorphism and we get a subset of $[0,1]$ homeomorphic to $\mathbb{Z}_p$.

Definition 3.1. A subset A of $\mathbb{R}^n$ homeomorphic to $\mathbb{Z}_p$ is called **Euclidean model** of $\mathbb{Z}_p$. In the special case $n = 1$, we call such subsets **linear models** of $\mathbb{Z}_p$.

Definition 3.2. A **fractal** is a self-similar geometric object. In other words, it can be split into parts, each of which is a reduced-size copy of the whole.

Definition 3.3. Informally speaking, the **self-similarity dimension** of a fractal is the statistical quantity that indicates how it appears to fill the space. More explicitely, if we take an object with euclidean dimension d and reduce its linear size by $\frac{1}{l}$ in each spacial dimension, then it takes $N = l^d$ self-similar objects to cover the original one. Thus

$$d = \log_l(N) = \frac{\ln(N)}{\ln(l)}.$$

Remark. Let us obtain this definition a little bit more formally: Let A be a d-dimensional fractal and let $E(A)$ denote its extent in the space (which is intuitively clear). Then $E(A)$ is a homogeneous function of degree d and A is the union of N translates of $\frac{1}{l}A$. Thus we get

$$E(A) = N \cdot E(\frac{1}{l}A) = \frac{N}{l^d}E(A).$$

Remark.

1. In the case of a linear model of $\mathbb{Z}_p$ (i.e. when $b > p$), we have de facto a fractal with $l = b$ and $N = p$. As one can imagine, $0 < d = \frac{\ln(p)}{\ln(b)} < 1$.
2. Note that

$$b \searrow p \qquad \Rightarrow \qquad d \nearrow 1.$$

Example 3.1.

1. The self-similarity dimension is a generalization of the usual dimension! Let A for example be a cube. If we reduce its linear size by $\frac{1}{l}$ in each spacial dimension then we obviously need l^3 little cubes to cover the original one. Thus

$$d = \frac{\ln(l^3)}{\ln(l)} = 3.$$

2. Let us take the Cantor set $C^{(p)}$ as in the previous section. Then

$$d = \frac{\ln(p)}{\ln(2p-1)}.$$

Note in particular that the usual Cantor set $C = C^{(2)}$ has dimension

$$d = \frac{\ln(2)}{\ln(3)} = 0.63.$$

3.2 Higher-dimensional Euclidean models of $\mathbb{Z}_p$

We want to generalize the map ψ as follows: in order to obtain an image in a general space V, we first need to map the digits of a p-adic number to some points in the space. Now the generalization becomes quite clear, since we will map this points in V under the map ψ.
In this subsection we will see how α and b has to be chosen to get a good generalization Ψ of the map ψ. But first let us define rigorously the map Ψ:

Definition 3.4.

1. A finitely dimensional inner product space V over $\mathbb{R}$ is called **Euclidean space**.

2. Let V be an Euclidean space and let $\nu : S := \{0, 1, \ldots, p-1\} \hookrightarrow V$ be an injective map. Then define $V \supset \Sigma := \nu(S)$ and

$$\Psi := \Psi_{\nu,b,p} : \mathbb{Z}_p \to V \qquad \sum_{i=0}^{\infty} a_i p^i \mapsto \alpha \sum_{i=0}^{\infty} \frac{\nu(a_i)}{b^{i+1}} \tag{2}$$

We already know we can express $\mathbb{Z}_p$ in a disjoint union as following:

$$\mathbb{Z}_p = \biguplus_{a \in S} a + p\mathbb{Z}_p.$$

Let us see what is the image of $\mathbb{Z}_p$ under Ψ:

$$\Psi(\mathbb{Z}_p) = \Psi(\biguplus_{a \in S} a + p\mathbb{Z}_p) = \bigcup_{a \in S} \Psi(a) + \Psi(p\mathbb{Z}_p) = \bigcup_{v \in \Sigma} \frac{\alpha v}{b} + \frac{1}{b}\Psi(\mathbb{Z}_p) \tag{3}$$

since multiply by p corresponds (under Ψ) to move the dot on the right in the numeration base b. Note the union is no more necessarily disjoint, but for b enough large it will be. In this case $\Psi(\mathbb{Z}_p)$ is injective and hence a homeomorphism (since continuity of Ψ and its inverse are obvious). Furthermore it is the union of disjoint self-similar images (i.e. a fractal) and hence we have an iterative construction of the spatial model.

Remark. As in the case for linear models, we need a good choice for α, in the sense that we want the smallest α such that all elements in Σ are reached by $\Psi(\mathbb{Z}_p)$.

Lemma 3.2. Choosing $\alpha = b-1$, $\Psi(\mathbb{Z}_p)$ remains in the convex hull $\bar{\Sigma}$ of Σ and hence it is the desired α.

Proof. Let $\lambda : V \to \mathbb{R}$ be an affine linear functional such that

$$\lambda \leq 1 \text{ on } \Sigma \text{ and } \lambda(v) = 1 \text{ for some } v \in \Sigma.$$

Then

$$\lambda(\alpha \sum_{i=0}^{\infty} \frac{\nu(a_i)}{b^{i+1}}) \leq \alpha \sum_{i=0}^{\infty} \frac{1}{b^{i+1}} = 1$$

and thus $F := \Psi(\mathbb{Z}_p) \subset \bar{\Sigma}$. □

To sum up, we obtain a good generalization Ψ of the homeomorphic map ψ if we choose $\alpha = b-1$ and b large enough in order to get a disjoint union in (3).

Before going to some clarifying examples let us see how we can obtain an iterative construction of a fractal through removing pieces:

Remark. The fractal $F := \Psi(\mathbb{Z}_p)$ is the intersection of a decreasing sequence of compact subsets K_n:

Let $K_0 := \bar{\Sigma}$. Then

$$F = \bigcup_{v \in \Sigma} \alpha \frac{v}{b} + \frac{F}{b} \subset \bigcup_{v \in \Sigma} \alpha \frac{v}{b} + \frac{K_0}{b} =: K_1$$

and inductively we have

$$F \subset K_n := \bigcup_{v \in \Sigma} \alpha \frac{v}{b} + \frac{K_{n-1}}{b} \text{ for all } n$$

This leads to the representation

$$F = \bigcap_{n=1}^{\infty} K_n$$

which is a lot useful for constructing inductively a fractal such as the Sierpinski gasket or for us an Euclidean model:

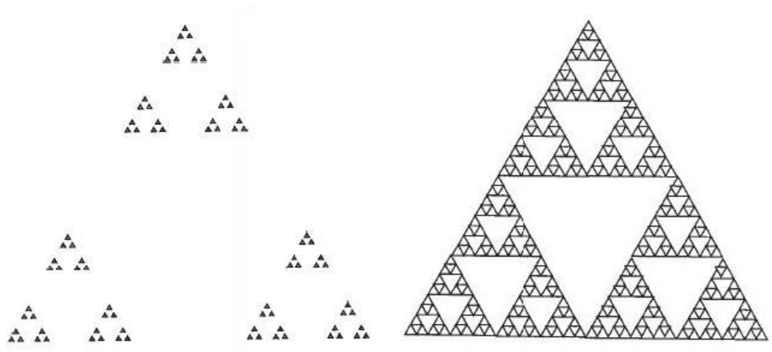

Figure 2: Model of $\mathbb{Z}_3$ and Sierpinski gasket

Example 3.2 (Sierpinski gasket). Let $p = 3$, $E = \mathbb{R}^2$ with basis $e_1 = (1, 0)$, $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b > 1$, $\alpha = b - 1$ (by the previous remark) and define

$$\nu(k) = \begin{cases} 0 & \text{if } k = 0 \\ e_1 & \text{if } k = 1 \\ e_2 & \text{if } k = 2 \end{cases}$$

Then

$$\Psi = \Psi_{3,b,\nu} : \mathbb{Z}_3 \to V \qquad \sum_{i=0}^{\infty} a_i 3^i \mapsto (b-1) \sum_{i=0}^{\infty} \frac{\nu(a_i)}{b^{i+1}}$$

We observe that the self-similar images are disjoint when $b > 2$. Thus in that case we get homeomorphic models of $\mathbb{Z}_3$ inside the triangle.
If $b = 2$ the image is the well-known Sierpinski gasket which is a connected figure (synonym of not injectivity of Ψ). For example we see that $-\frac{3}{2}$ and 1 have the same image under Ψ: first note $-\frac{3}{2} = \frac{1}{1-3} - 1 = \sum_{i=1}^{\infty} 3^i$. Hence

$$\Psi(-\frac{3}{2}) = \sum_{i=2}^{\infty} \frac{e_1}{2^i} = e_1 \left(\frac{1}{1 - \frac{1}{2}} - 1 - \frac{1}{2} \right) = \frac{e_1}{2} = \Psi(1)$$

In general, taking the enumeration base at the 'limit' of injectivity gives connected fractals which are the well-know ones!

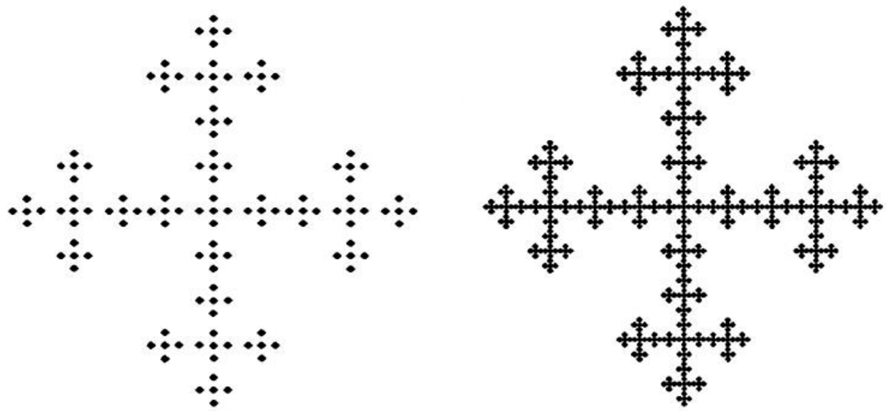

Figure 3: Model of $\mathbb{Z}_5$ and limiting fractal

Example 3.3. Let $p = 5$, $E = \mathbb{R}^2$ and define

$$\nu(k) = \begin{cases} (0,0) & \text{if } k = 0 \\ (1,0) & \text{if } k = 1 \\ (0,1) & \text{if } k = 2 \\ (-1,0) & \text{if } k = 3 \\ (0,-1) & \text{if } k = 4 \end{cases}$$

We want to discuss injectivity: because of the symmetry it is enough to check it for the points lying between $(0,0)$ and $(1,0)$. We want to avoid that the two p-adic integers $\sum_{i=1}^{\infty} 5^i$ and $1 + \sum_{i=1}^{\infty} 3 \cdot 5^i$ have the same image. It is easy to see this happens when $b = 3$ since the two vertices of the squares meet if $\Psi(1) = \frac{b-1}{b} = \frac{2}{3}$. As above the limit case $b = 3$ gives a connected fractal but not a homeomorphic model of $\mathbb{Z}_5$.

Example 3.4 (Generalization of Example 3.3). Let $p > 3$, $E = \mathbb{R}^2$ and $\alpha = b - 1$. We can generate a regular $(p-1)$-gon centered in 0 with vertices $e_1, \ldots, e_{p-1}$. Let us choose them with norm equal 1 and define $\nu(k) = e_k$ for $k \neq 0$ and $\nu(0) = 0$. Then the injectivity of the map Ψ is obtained by b large enough: more precisely, for $p \geq 7$ the minimal distance between points in $(\Psi(k))_{k \in S}$ is between $\Psi(k)$ and $\Psi(k+1)$, $k \notin 0, p-1$, and it is given by

$\frac{2(b-1)}{b}\sin(\frac{\pi}{p-1})$. Then b has to satisfy

$$(b-1)\sum_{i=2}^{\infty}\frac{1}{b^i}=\frac{1}{b}<\frac{b-1}{b}\sin(\frac{\pi}{p-1})$$

and this gives the criterium $b>\frac{1}{\sin(\frac{\pi}{p-1})}+1$ for injectivity. Note that $p=5$ has the same b of $p=7$, since the minimal distance between points in Σ is between the center 0 and a vertice e_k and for $p=7$ we get an exagon where the minimal distance between points in Σ is both the distance between the center 0 and a vertice e_k and the one between two vertices next to each other.

Remark. Let us now calculate the dimension of the fractals obtained by letting b to the limit of injectivity of the map Ψ we have seen in this subsection. The dimension of the Sierpinski gasket is

$$d=\frac{\ln(3)}{\ln(2)}=1.58,$$

the one of the limiting quadratic fractal in Example 3.3 is

$$d=\frac{\ln(5)}{\ln(3)}=1.46$$

and the dimension of the fractal in the $(p-1)$-gon is

$$d=\begin{cases}\frac{\ln(7)}{\ln(3)}=1.77 & \text{if } p=5\\ \frac{\ln(p)}{\ln\left(\frac{1}{\sin\left(\frac{\pi}{p-1}\right)}+1\right)} & \text{if } p\geq 7\end{cases}$$

e.g. for $p=83$ $d=1.339$.

3.3 Euclidean models of $\mathbb{Q}_p$

We know that we can express $\mathbb{Q}_p$ as

$$\mathbb{Q}_p=\bigcup_{m\geq 0}p^{-m}\mathbb{Z}_p. \tag{4}$$

But in the previous section we saw that a model of $p\mathbb{Z}_p$ is a contraction of ratio $\frac{1}{b}$ of the model $\mathbb{Z}_p$. So conversely a dilatation of ratio b of the model $\mathbb{Z}_p$

gives us a model of $\frac{1}{p}\mathbb{Z}_p$. Thus inductively by (4) an euclidean model of $\mathbb{Q}_p$ is nothing else but an extension of the model of $\mathbb{Z}_p$ to the whole Euclidean space V. As one can imagine, the homeomorphism giving the Euclidean model is

$$\Psi := \Psi_{\nu,b,p} : \mathbb{Q}_p \to V \qquad \sum_{i=-\infty}^{\infty} a_i p^i \mapsto \alpha \sum_{i=-\infty}^{\infty} \frac{\nu(a_i)}{b^{i+1}} \tag{5}$$

with the same assumptions as for equation (2).
Let us explain this more carefully for the model constructed in Example 3.4: take the $(p-1)$-gon which contains the fractal of $\mathbb{Z}_p$ and copy it to its $(p-1)$ vertices such that the two opposite vertices touch each other; we thus obtain a new much bigger $(p-1)$-gon containing p fractals and this will be the model of $\frac{1}{p}\mathbb{Z}_p = \{\zeta \in \mathbb{Q}_p \mid a_{-k} = 0 \text{ for } k \geq 2\}$. Inductively, take the $(p-1)$-gon containing the model of $\frac{1}{p^{n-1}}\mathbb{Z}_p$, copy it to its $(p-1)$ vertices and obtain a model of $\frac{1}{p^n}\mathbb{Z}_p$.

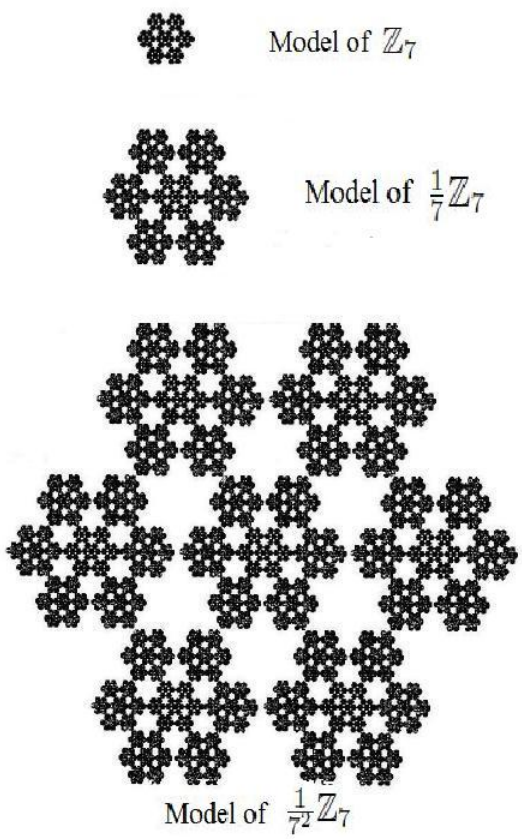

Figure 4: Iterative construction of the model of $\mathbb{Q}_7$ given the one of $\mathbb{Z}_7$