1. Another combinatorial nonimplication

We will prove that Equation 1076, which is $x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$, does not generally imply Equation 3, which is $x = x \diamond x$. It was more convenient for me to work with a linearization of the form $x \diamond y = x + f(h)$ where $h = y - x$.

Working out the linearization of Equation 1076, we find

$$\begin{aligned} x &= y \diamond ((x \diamond [x + f(h)]) \diamond y) \\ &= y \diamond ([x + f^2(h)] \diamond y) \\ &= y \diamond [x + f^2(h) + f(h - f^2(h))] \\ &= x + h + f(f(h - f^2(h)) - (h - f^2(h))), \end{aligned}$$

and so we want to choose f satisfying the functional equation

$$f[f(h - f^2(h)) - (h - f^2(h))] = -h.$$

On the other hand, the linearization of Equation 3 leads us to want $f(0) \neq 0$.

We take our carrier set to be a free abelian group A on $\aleph_0$ generators. (**Note**: One could just as well take $\aleph_1$ generators, if the initial seed needed to be infinite.)

Let $\mathscr{E}$ be the collection of sets $E \subseteq A^2$ where the following properties holds.

(1) E is finite.
(2) E is a function (i.e., no element of A occurs as the first coordinate of two different ordered pairs in E).
(3) There are enough pairs in E so that treating E as a partial definition of f, the functional equation holds when $h = 0$.
(4) If $(a, b), (b, c) \in E$, then $(a - c, d), (-a + c + d, -a) \in E$ for some $d \in A$. (Thus, once we know that E defines both $f(a)$ and $f^2(a)$, the functional equation holds when $h = a$.)

We partially order $\mathscr{E}$ under set inclusion. The set $\mathscr{E}$ is nonempty, by noting $\{(0, 0)\} \in \mathscr{E}$. Alternatively, taking $p, q, r, s \in A$ to be independent generators, one can quickly check that

$$E_0 := \{(0, p), (p, q), (-q, r), (q + r, 0), (-p + q + r, s), (p - q - r + s, -q - r)\}$$

also belongs to $\mathscr{E}$. We will primarily work with extensions of E_0 in $\mathscr{E}$, in order to guarantee that $f(0) \neq 0$.

Lemma 1.1. *For each $E \in \mathscr{E}$ and for each $z \in A$, there is an extension $E \subseteq E' \in \mathscr{E}$ where E' satisfies the functional equation when $h = z$.*

Proof. **Case 1**: Assume that z is the first coordinate of some pair in E.

Fix b to be the (unique, by condition (2)) second coordinate. If b is the first coordinate of some pair in E, then by condition (4), the functional equation already holds for z by taking $E' = E$. So, for the rest of this case, assume that b is not a first coordinate of any pair.

Since E is finite by (1), b appears as the second coordinate of only finitely many pairs. Fix $a_1, \ldots, a_n$ to be the list of the distinct first coordinates for such pairs. (So one of the a_i is z.) Note that none of these a_i is zero, by condition (3), since $f^2(a_i)$ is not defined.

For each integer $i \in [1, n]$, the element $-a_i \in A$ can appear as the first coordinate in at most one ordered pair in E. Let S be the set of indices where there is such a pair, and for each $i \in S$ fix b_i' to be the unique element where $(-a_i, b_i') \in E$. Let $T = \{i \in S : b_i' = 0\}$.

Now, let c, d_i $(i \in [1,n])$, and e_j $(j \in S-T)$ be a set of independent generators of A that do not appear anywhere in the support of E. We take

$$\begin{aligned} E' \ &:= \ E \cup \{(b,c)\} \cup \{(a_i - c, d_i), (-a_i + c + d_i, -a_i)\}_{i\in[1,n]} \\ &\quad \cup \{(-a_j - b'_j + c + d_j, e_j), (a_j + b'_j - c - d_j + e_j, a_j - c - d_j)\}_{j\in S-T} \\ &\quad \cup \{(-c - d_k, a_k - c - d_k)\}_{k\in T}. \end{aligned}$$

A finite check verifies each of the conditions, showing that $E' \in \mathscr{E}$ and that E' satisfies the functional for each a_i (hence for z).

Case 2: Assume that z is not the first coordinate of some pair in E.

If z appears as the second coordinate of some pair in E, say (z', z), by applying Case 1 to z', we can extend E to a new set F so that the functional equation holds for z'. Now, z appears as a first coordinate of a pair in F, so using Case 1 again we can extend F so that the functional equation holds for z.

If z doesn't appear as either a first or a second coordinate, then by adding a new pair (z, z'), where z' is any element of A not appearing in the support of E, we again reduce to Case 1. □

We are now essentially done. Well-ordering A, with order type ω, we may recursively guarantee that there is an ω chain in $\mathscr{E}$, with E_0 at the bottom, where each element of A becomes a first coordinate somewhere in the chain. The union of such a chain is the needed complete f.