Spectra of equational laws

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based on discussions of the Equational Theories Project

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In these notes we discuss four problems related to the spectrum of a law, namely to the set of cardinalities of (non-empty) models of the law. For instance, a law that is equivalent to law 2 ($x = y$) has spectrum $\{1\}$. Since the direct product of magmas satisfying a law also does so, the spectrum is stable under multiplication, and includes all infinite cardinalities as soon as it contains any value > 1. We thus focus on finite cardinalities and discuss the spectrum as being a multiplicative subset of $\mathbb{Z}_{>0}$. The four problems of interest are as follows.

- In section 1, finding which laws have full spectrum $\mathbb{Z}_{>0}$.
- In section 2, finding the spectrum when it is not $\mathbb{Z}_{>0}$.
- In section 3, finding the finite cardinalities of directly-irreducible, subdirectly-irreducible, and simple magmas satisfying a law.
- In section 4, counting magmas of a given cardinality satisfying a law. This leads to a notion of phases (vacuum, gas, liquid, solid, crystal).

We perform these investigations for laws up to different orders depending on the problem, as they present very different challenges.

Several open questions seem within reach of a short dedicated effort.

- Build magmas obeying law 115 $x = y \diamond ((x \diamond x) \diamond y)$ (and as a result law 873) of all cardinalities $n \equiv 2 \mod 3$ other than $n = 2$, see subsection 2.2.
- Build magmas obeying law 1489 based on a graph theory description, see subsubsection 2.5.1.
- Build enough semi-symmetric quasigroups (magmas obeying law 14) to saturate the upper bound even when the 2-adic valuation of n is low, see subsection 4.2. Treat analogously law 66.
- Generalize the construction that shows that law 1682 has full spectrum, to deal with some higher-order laws whose spectrum is not yet proven to be full, see section 1.

Several points remain to be done but are rather tedious or computationally intensive.

- Finish counting magmas for laws of order 3 of other shapes.
- Add some more terms to the start of the spectrum for laws whose spectrum is yet unknown, to help formulate a conjecture.

Other questions seem harder.

- Find the lowest-numbered law that features models of cardinality 2 and 3, but no model of cardinality 5.
- Find the spectrum of the Dupont law 63 $x = y \diamond (x \diamond (x \diamond y))$, and use the resulting techniques to resolve other equations in subsection 2.3. Even without knowing the spectrum it might be possible to lower-bound the number of magmas obtained by cohomological constructions.

Table 1: Number of laws of order 0–9 whose spectrum is known to be full (by linear or piecewise or ad-hoc models), or non-full (by the absence of model of size 2 or 3). Out of 5 996 643 396 laws of order up to 9, 3 607 005 194 (60.150%) have full spectrum, 2 389 459 735 (39.847%) do not, and 178 467 (0.003%) remain unknown.

Order	Full spectrum				Non-full spectrum			Unknown
	total =	linear +	pieces +	...	total =	no size 2 +	no size 3	
0	1 =	1 +	0 +	0	1 =	1 +	0	0
1	3 =	3 +	0 +	0	2 =	2 +	0	0
2	27 =	27 +	0 +	0	12 =	12 +	0	0
3	229 =	229 +	0 +	0	135 =	135 +	0	0
4	2814 =	2808 +	4 +	2	1470 =	1408 +	62	0
5	35950 =	35916 +	32 +	2	21932 =	21920 +	12	0
6	558125 =	557137 +	978 +	10	329940 =	315280 +	14660	300
7	9227082 =	9223929 +	3153 +	0	5892466 =	5889143 +	3323	557
8	172221522 =	172064190 +	157332 +	0	109694882 =	105709286 +	3985596	25086
9	3424959441 =	3424347706 +	611735 +	0	2273518895 =	2272623385 +	895510	152524

1 Laws with a full spectrum

We explain here in a streamlined way the results of the investigation discussed at https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Equations.20with.20full.20spectrum/near/489967644. The results are summarized in Table 1.

1.1 Laws up to order 4

Consider laws of order up to 4, modulo duality and equivalence, and denote these 741 classes by the lowest-numbered equation. We list them in Table 2 for reference. All laws obeyed by constant laws, or left/right projection, or by $x \diamond y = \pm x \pm y$ on $\mathbb{Z}/n\mathbb{Z}$ have full spectrum. This leaves 48 laws (and their duals and equivalents) that are not given a full spectrum by these considerations:

[2, 63, 66, 73, 115, 118, 125, 167, 168, 467, 474, 481, 501, 546, 556, 667, 670, 677, 695, 704, 873, 880, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1323, 1480, 1482, 1483, 1485, 1486, 1489, 1496, 1516, 1523, 1526, 1682, 1685, 1692, 1719]

For laws 1482, 1523, 1682 we have found models of all sizes:

- for law 1482, the model found by Douglas McNeil is $x \diamond y = 0$ except $0 \diamond 0 = 1$ and $0 \diamond x = x \diamond 0 = x$ for $x \neq 0, 1$;
- for law 1523, $x \diamond x = 0$, $x \diamond 0 = 0 \diamond x = x$.
- for law 1682, the model found by Zoltan Kocsis on any interval $[0, n)$ is

Table 2: List of representatives of equivalence classes modulo duality.

[1, 2, 3, 4, 8, 9, 10, 11, 13, 14, 16, 38, 40, 41, 43, 47, 48, 49, 50, 52, 53, 55, 56, 58, 62, 63, 65, 66, 72, 73, 75, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 111, 115, 117, 118, 124, 125, 127, 138, 151, 152, 153, 159, 162, 167, 168, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 323, 325, 326, 327, 329, 332, 333, 335, 336, 343, 411, 412, 413, 414, 416, 417, 418, 419, 420, 422, 426, 427, 428, 429, 430, 432, 433, 434, 436, 437, 439, 440, 442, 443, 446, 450, 452, 455, 463, 464, 466, 467, 473, 474, 476, 477, 481, 492, 500, 501, 503, 504, 508, 510, 511, 513, 543, 546, 556, 562, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 626, 629, 630, 632, 633, 635, 639, 640, 642, 643, 645, 646, 647, 653, 655, 657, 658, 667, 669, 670, 676, 677, 679, 680, 690, 692, 695, 703, 704, 706, 707, 713, 714, 716, 723, 727, 731, 765, 778, 817, 818, 819, 820, 822, 823, 824, 825, 826, 827, 828, 829, 832, 833, 834, 835, 836, 837, 838, 839, 840, 842, 843, 844, 845, 846, 847, 848, 854, 856, 860, 861, 870, 872, 873, 879, 880, 882, 883, 887, 895, 898, 906, 907, 910, 916, 917, 947, 960, 978, 1020, 1021, 1022, 1023, 1025, 1026, 1027, 1028, 1029, 1032, 1033, 1035, 1036, 1037, 1038, 1039, 1041, 1042, 1043, 1045, 1046, 1048, 1049, 1050, 1051, 1052, 1053, 1055, 1056, 1060, 1061, 1063, 1073, 1075, 1076, 1082, 1083, 1085, 1086, 1096, 1109, 1110, 1112, 1113, 1117, 1119, 1122, 1133, 1137, 1167, 1171, 1184, 1223, 1224, 1225, 1226, 1227, 1228, 1229, 1230, 1231, 1232, 1233, 1234, 1235, 1236, 1237, 1238, 1239, 1240, 1241, 1242, 1243, 1244, 1245, 1248, 1249, 1250, 1251, 1252, 1253, 1254, 1255, 1256, 1259, 1262, 1263, 1264, 1267, 1271, 1276, 1278, 1279, 1285, 1286, 1288, 1289, 1312, 1313, 1315, 1316, 1322, 1323, 1325, 1340, 1353, 1370, 1374, 1387, 1426, 1427, 1428, 1429, 1431, 1432, 1434, 1435, 1437, 1441, 1442, 1443, 1444, 1445, 1446, 1447, 1448, 1451, 1453, 1454, 1457, 1461, 1465, 1469, 1478, 1479, 1480, 1481, 1482, 1483, 1484, 1485, 1486, 1488, 1489, 1491, 1492, 1496, 1515, 1516, 1518, 1519, 1523, 1525, 1526, 1586, 1629, 1630, 1631, 1632, 1633, 1634, 1635, 1636, 1637, 1638, 1641, 1644, 1645, 1647, 1648, 1650, 1654, 1655, 1657, 1660, 1664, 1672, 1681, 1682, 1684, 1685, 1687, 1691, 1692, 1694, 1695, 1701, 1718, 1719, 1721, 1722, 1724, 1728, 1729, 1731, 1738, 1793, 3253, 3254, 3255, 3256, 3257, 3258, 3259, 3260, 3261, 3262, 3263, 3264, 3265, 3267, 3268, 3269, 3270, 3271, 3272, 3273, 3274, 3275, 3277, 3278, 3279, 3280, 3281, 3282, 3283, 3284, 3285, 3288, 3290, 3292, 3294, 3296, 3297, 3300, 3306, 3308, 3309, 3312, 3315, 3316, 3317, 3318, 3319, 3320, 3321, 3322, 3323, 3326, 3331, 3334, 3342, 3343, 3345, 3346, 3349, 3350, 3352, 3353, 3355, 3363, 3364, 3385, 3388, 3398, 3414, 3417, 3456, 3457, 3458, 3459, 3460, 3461, 3462, 3463, 3464, 3465, 3466, 3467, 3468, 3469, 3470, 3471, 3472, 3473, 3474, 3475, 3476, 3477, 3478, 3479, 3480, 3481, 3482, 3483, 3484, 3485, 3487, 3488, 3489, 3491, 3493, 3495, 3496, 3497, 3499, 3503, 3509, 3511, 3512, 3513, 3515, 3518, 3519, 3520, 3521, 3522, 3523, 3524, 3525, 3526, 3527, 3529, 3532, 3533, 3534, 3537, 3541, 3545, 3546, 3548, 3549, 3555, 3556, 3558, 3566, 3583, 3587, 3588, 3591, 3600, 3601, 3607, 3617, 3620, 3634, 3659, 3660, 3661, 3662, 3663, 3665, 3666, 3667, 3668, 3669, 3670, 3671, 3672, 3673, 3675, 3676, 3678, 3679, 3681, 3682, 3683, 3703, 3712, 3714, 3715, 3716, 3718, 3721, 3722, 3723, 3724, 3726, 3727, 3728, 3729, 3730, 3735, 3737, 3740, 3744, 3748, 3751, 3756, 4268, 4269, 4270, 4271, 4272, 4273, 4274, 4275, 4276, 4277, 4278, 4279, 4280, 4283, 4284, 4286, 4287, 4288, 4290, 4291, 4293, 4295, 4296, 4297, 4299, 4300, 4301, 4303, 4304, 4305, 4314, 4315, 4318, 4320, 4321, 4325, 4327, 4331, 4343, 4358, 4362, 4364, 4369, 4380, 4381, 4382, 4383, 4384, 4385, 4386, 4387, 4388, 4389, 4390, 4391, 4392, 4393, 4396, 4397, 4398, 4399, 4400, 4401, 4402, 4403, 4404, 4405, 4406, 4407, 4408, 4410, 4411, 4412, 4413, 4415, 4416, 4417, 4421, 4423, 4424, 4428, 4430, 4433, 4434, 4435, 4437, 4438, 4439, 4441, 4443, 4444, 4445, 4447, 4448, 4449, 4456, 4458, 4460, 4461, 4470, 4471, 4474, 4476, 4478, 4481, 4482, 4484, 4485, 4490, 4497, 4502, 4512, 4513, 4515, 4517, 4519, 4520, 4526, 4531, 4535, 4541, 4544]

defined by

$$i \diamond j = \begin{cases} i & \text{if } j = 0, \\ \delta_{i=0} & \text{if } j = 1, \\ 1 + \delta_{j \text{ odd}} & \text{if } i = 0 \text{ and } j \geq 2, \\ j & \text{if } i = 1 \text{ and } j \geq 3 \text{ odd}, \\ i + \delta_{i<j} & \text{if } i, j \geq 2 \text{ and } i + j \text{ even}, \\ 2\delta_{i=1} + 0 & \text{if } i \geq 1 \text{ odd and } j = 2, \\ j - \delta_{i \geq j} & \text{if } i \geq 1 \text{ and } j \geq 3 \text{ and } i + j \text{ odd}. \end{cases} \tag{1}$$

For the other laws in the list, we show

- The 31 laws [2, 63, 66, 73, 115, 118, 125, 167, 168, 467, 474, 501, 670, 677, 704, 873, 880, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1480, 1486, 1489, 1516, 1685, 1692, 1719] have no model of size 2.
- The 14 laws [481, 546, 556, 667, 695, 883, 887, 895, 898, 1323, 1483, 1485, 1496, 1526] have models of size 2 but none of size 3.

These $3+31+14 = 48$ laws cover the whole list above. Curiously, this situation persists to higher orders: laws that have models of size 2 and 3 appear to have full spectrum in all cases we were able to resolve.

1.2 Higher-order laws todo

https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Equations.20with.20full.20spectrum/near/489967644

2 Spectrum of laws of order up to 4

We explain here in a streamlined way the results of the investigation discussed at https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Order.203.20Spectra/near/526300502. From the full-spectrum discussion, we already know that many laws have full spectrum, either through linear models, or through ad-hoc models for laws 1482, 1523, 1682. This leaves 45 laws up to duality and equivalence:

2	$x = y$	667	$x = y \diamond (x \diamond ((x \diamond x) \diamond y))$	1279	$x = y \diamond (((x \diamond x) \diamond y) \diamond y)$
63	$x = y \diamond (x \diamond (x \diamond y))$	670	$x = y \diamond (x \diamond ((x \diamond y) \diamond y))$	1286	$x = y \diamond (((x \diamond y) \diamond x) \diamond y)$
66	$x = y \diamond (x \diamond (y \diamond y))$	677	$x = y \diamond (x \diamond ((y \diamond x) \diamond y))$	1313	$x = y \diamond (((y \diamond x) \diamond x) \diamond y)$
73	$x = y \diamond (y \diamond (x \diamond y))$	695	$x = y \diamond (x \diamond ((z \diamond z) \diamond y))$	1323	$x = y \diamond (((y \diamond y) \diamond x) \diamond y)$
115	$x = y \diamond ((x \diamond x) \diamond y)$	704	$x = y \diamond (y \diamond ((x \diamond x) \diamond y))$	1480	$x = (y \diamond x) \diamond (x \diamond (x \diamond z))$
118	$x = y \diamond ((x \diamond y) \diamond y)$	873	$x = y \diamond ((x \diamond x) \diamond (y \diamond y))$	1483	$x = (y \diamond x) \diamond (x \diamond (y \diamond z))$
125	$x = y \diamond ((y \diamond x) \diamond y)$	880	$x = y \diamond ((x \diamond y) \diamond (x \diamond y))$	1485	$x = (y \diamond x) \diamond (x \diamond (z \diamond y))$
167	$x = (y \diamond x) \diamond (x \diamond y)$	883	$x = y \diamond ((x \diamond y) \diamond (y \diamond y))$	1486	$x = (y \diamond x) \diamond (x \diamond (z \diamond z))$
168	$x = (y \diamond x) \diamond (x \diamond z)$	887	$x = y \diamond ((x \diamond y) \diamond (z \diamond z))$	1489	$x = (y \diamond x) \diamond (y \diamond (x \diamond y))$
467	$x = y \diamond (x \diamond (x \diamond (y \diamond y)))$	895	$x = y \diamond ((x \diamond z) \diamond (y \diamond z))$	1496	$x = (y \diamond x) \diamond (y \diamond (z \diamond z))$
474	$x = y \diamond (x \diamond (y \diamond (x \diamond y)))$	898	$x = y \diamond ((x \diamond z) \diamond (z \diamond y))$	1516	$x = (y \diamond y) \diamond (x \diamond (x \diamond y))$
481	$x = y \diamond (x \diamond (y \diamond (z \diamond z)))$	907	$x = y \diamond ((y \diamond x) \diamond (x \diamond y))$	1526	$x = (y \diamond y) \diamond (y \diamond (x \diamond y))$
501	$x = y \diamond (y \diamond (x \diamond (x \diamond y)))$	1076	$x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$	1685	$x = (y \diamond x) \diamond ((x \diamond y) \diamond y)$
546	$x = y \diamond (z \diamond (x \diamond (z \diamond y)))$	1083	$x = y \diamond ((x \diamond (y \diamond x)) \diamond y)$	1692	$x = (y \diamond x) \diamond ((y \diamond x) \diamond y)$
556	$x = y \diamond (z \diamond (y \diamond (x \diamond z)))$	1110	$x = y \diamond ((y \diamond (x \diamond x)) \diamond y)$	1719	$x = (y \diamond y) \diamond ((x \diamond x) \diamond y)$

Some of these laws are equivalent for finite magmas, specifically laws [63,1692], laws [73,118*,125], laws [115,880*], laws [481,1496*], laws [883*,1323,1526] where * denotes the dual law. This reduces the number of laws to 38.

Many of these laws (for instance those of the form $x = y\diamond\dots$) imply that left multiplications are surjective. In a finite setting the left multiplications are thus bijective, namely the magma is a left quasigroup. There is then a left division operation uniquely defined by $x \diamond (x : y) = y$. For several pairs of laws, "$\diamond$" obeys one of the laws if and only if ":" obeys the other (or its dual), in other words the laws are parastrophically equivalent in finite magmas.[1] This means that the two equations share the same finite spectrum.[2] In this way we find

$$\mathrm{spec}(\text{E63}) = \mathrm{spec}(\text{E73}), \qquad \mathrm{spec}(\text{E546}) = \mathrm{spec}(\text{E556}). \tag{2}$$

Note that in a (left-)quasigroup obeying law 467 $x = y \diamond (x \diamond (x \diamond (y \diamond y)))$, the left division operation defined by $x \diamond (x : y) = y$ obeys law 437 $x = x : (y :$

[1]Normally parastrophic equivalence is defined for quasigroups and allows for both left and right division, and for the operations with operands swapped. In the context of left quasigroups, right division is not defined so only four of the six parastrophes are allowed.

[2]When discussing the (sub)directly irreducible and simple spectra there could be differences for infinite cardinalities, because the laws typically do not imply that infinite magmas are left quasigroups.

$(y : (x : y)))$, but we cannot make use of that because the models of law 437 that we used to show the full spectrum property are left-projections, which are maximally far from being left quasigroups. The same problem occurs for law 481. Laws 474 and 501 and 898 are parastrophically equivalent to themselves (or their dual). The other laws do not have a shape conducive to having a parastrophically equivalent law. We are left with 36 laws which could have distinct spectra, distinct from $\mathbb{Z}_{>0}$:

[2, 63, 66, 115, 167, 168, 467, 474, 481, 501, 546, 667, 670, 677, 695, 704, 873, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1480, 1483, 1485, 1486, 1489, 1516, 1685, 1719]

We organize the discussion around five classes of laws: related to abelian groups, to semi-symmetric quasigroups, to the Dupont law 63, to the central groupoid law 168, and some isolated laws. Before we start, we note the obvious fact

$$\operatorname{spec}(\text{E2}) = \{1\}. \tag{3}$$

2.1 Laws from which a group structure is definable

First some laws related to abelian groups. Note that there are some more laws implying linearity, which have already been eliminated at an earlier stage by models of the form $x \diamond y = \pm x \pm y$, most crucially Tarski's law 543 defining abelian group subtraction.

546 $x = y \diamond (z \diamond (x \diamond (z \diamond y)))$ (and 556) has spectrum $\{k^2 + l^2 \mid k, l \in \mathbb{Z}\}$, namely all integers $n \geq 1$ whose valuation $v_p(n)$ is even for each prime $p \equiv 3 \mod 4$. For any fixed element u, the operation $x \ominus_u y := (x \diamond u) \diamond (u \diamond y)$ obeys the Tarski law 543, namely is group subtraction. With some more work one can show that $x \diamond y = -x + \iota(y) + c$ in some module over $\mathbb{Z}[\sqrt{-1}]$ with ι being multiplication by $\sqrt{-1}$. Thus the spectrum is that of abelian groups such that there exists an automorphism ι that squares to negation. Finite abelian groups are products of p-groups

$$M = \prod_{p \text{ prime}} M_p, \qquad M_p = \prod_{j=1}^{l_p} \mathbb{Z}/p^{m_j}\mathbb{Z}, \tag{4}$$

and the automorphism ι acts independently as automorphisms $\iota_p \colon M_p \to M_p$. Since $(\iota_p)^{\circ 4} = \mathrm{id}$, the orbits of ι_p have 4, 2, or 1 element. For $p \neq 2$, the square $\iota_p \circ \iota_p$ has a unique fixed point $0 \in M_p$ so all other orbits of ι_p have 4 elements and thus $|M_p| = p^{v_p(n)} \equiv 1 \mod 4$. This shows that $v_p(n)$ is even for any prime $p \equiv 3 \mod 4$. Conversely, constructing magmas for each value in the spectrum is immediate: $\mathbb{Z}[\sqrt{-1}]/(k + \sqrt{-1}l)\mathbb{Z}[\sqrt{-1}]$ has $k^2 + l^2$ elements.

895 $x = y \diamond ((x \diamond z) \diamond (y \diamond z))$ has spectrum $\{2^n \mid n \geq 0\}$. It characterizes Boolean groups (abelian groups of exponent 2).

898 $x = y \diamond ((x \diamond z) \diamond (z \diamond y))$ has spectrum $\{2^n \mid n \geq 0\}$. It is at least that because law 895 implies 898. To get the converse, we work out that the law 898 implies, for any fixed element u, that the operation $x \oplus_u y = ((u \diamond x) \diamond (y \diamond u)) \diamond u$ obeys law 895.

2.2 Laws related to semi-symmetric quasigroups 14

Next, let us discuss laws related to the semi-symmetric quasigroup law 14. Here, we call Mendelsohn quasigroup an idempotent semi-symmetric quasigroup (obeying law 4961, $x = y \diamond (x \diamond (y \diamond (z \diamond (y \diamond z))))$, or equivalently laws 3 and 14). They are in one-to-one correspondence with Mendelsohn triple systems on the same set so their spectrum is $\{0, 1 \mod 3\} \setminus \{6\}$.

Semi-symmetric loops (magmas obeying law 14 and 40) are in one-to-one correspondence with Mendelsohn triple systems on $M \setminus \{e\}$ so their spectrum is $\{1, 2 \mod 3\} \setminus \{7\}$.

66 $x = y \diamond (x \diamond (y \diamond y))$ has spectrum $\{0, 1 \mod 3\} \setminus \{6\}$. Such magmas are in one-to-one correspondence with Mendelsohn quasigroups equipped with an involutive automorphism. In one direction, denote $S(x) = x \diamond x$ and verify that the operation $x \Box y = S(x \diamond y)$ obeys laws 3 and 14, and that $S(S(x)) = x$ and $S(x \diamond y) = S(x) \diamond S(y)$. In the other direction, verify that $x \diamond y = S(x \Box y)$ satisfies law 66 provided "$\Box$" obeys laws 3 and 14 and S is an involutive automorphism. The spectrum is thus contained in that of Mendelsohn quasigroups, and since S can be taken to be the identity the spectra are equal.

115 $x = y \diamond ((x \diamond x) \diamond y)$ (and 880) has **conjectural** spectrum $[1, +\infty) \setminus \{2, 6\}$. The law is obeyed in Mendelsohn quasigroups so its spectrum contains all $n \equiv 0, 1 \mod 3$ except 6. There remains to prove existence of models of all sizes $n = 3k + 2 \geq 5$ (showing it for $n = 8$, n prime, and $n/2$ prime would be enough). An ATP run shows there are models at least up to order 53, in which the squaring map $S \colon x \mapsto x \diamond x$ has a large cycle of size $3k + 1$ (or possibly $3j + 1$ for j close to k), making most of the magma translationally-invariant. For magma sizes up to 14, the possible cycle sizes for the squaring map (apart from having it be the identity) are $5 = 4+1$, $7 = 7$, $8 = 7+1$, $9 = 6+1+1+1$, $10 = 7+1+1+1$, $11 = 5+5+1$, $11 = 7+1+1+1+1$, $12 = 8+1+1+1+1$, $12 = 9+1+1+1$, $13 = 4+4+4+1$, $13 = 8+4+1$, $13 = 9+1+1+1+1$, $13 = 13$, $14 = 5+5+1+1+1+1$, $14 = 7+7$, $14 = 10+1+1+1+1$, $14 = 12+1+1$. There is probably some counting argument for which sizes are possible.

481 $x = y \diamond (x \diamond (y \diamond (z \diamond z)))$ (and 1496) has **conjectural** spectrum $[1, +\infty) \setminus \{3, 6\}$. It is equivalent to unipotence $x \diamond x = e$ and $x = y \diamond (x \diamond (y \diamond e))$. Finite magmas are quasigroups and the maps $x \mapsto x \diamond e$ and $x \mapsto e \diamond x$ are automorphisms, inverses of each other. The special case where these automorphisms are the identity is semi-symmetric loops. The spectrum thus contains $\{1, 2 \mod 3\} \setminus \{7\}$. It also contains $\{7, 9, 12\}$ by an ATP search. We see in these examples that the squaring map has specific cycle sizes. A natural conjecture is that the law is flexible enough to allow arbitrary magma sizes, except for the low-lying values 3,6.

501 $x = y \diamond (y \diamond (x \diamond (x \diamond y)))$ has **unknown** spectrum containing 1,4,5,8,9. It implies that the operation $x \Box y = x \diamond (x \diamond y)$ obeys law 14.

667 $x = y\diamond(x\diamond((x\diamond x)\diamond y))$ has **unknown** spectrum containing $\{1,2 \mod 3\}\cup\{9\} = \{1,2,4,5,7,8,9,10,11,\dots\}$. It is obeyed by semi-symmetric loops (hence the spectrum contains $\{1,2 \mod 3\}\setminus\{7\}$, and by idempotent magmas satisfying the Dupont law 63 (see below), from which one gets 7 in the spectrum. An ATP run gives that 9 is also there.

695 $x = y \diamond (x \diamond ((z \diamond z) \diamond y))$ has spectrum $\{1,2 \mod 3\} \setminus \{7\}$. Indeed, any semi-symmetric loop obeys law 695, and conversely the operation $x\Box y = (x \diamond x) \diamond (x \diamond y)$ obeys the semi-symmetric loop law 887.

873 $x = y \diamond ((x \diamond x) \diamond (y \diamond y))$ **conjectural** spectrum $[1,+\infty) \setminus \{2,6\}$. The spectrum contains that of law 115 because the dual of law 115 implies law 873. The spectrum does not contain 2 and 6 thanks to an ATP run.

883 $x = y\diamond((x\diamond y)\diamond(y\diamond y))$ (and 1323, 1526) has **unknown** spectrum containing $\{1,2 \mod 3\}$. It is obeyed by semi-symmetric loops *and* idempotent magmas obeying law 73 (parastrophically equivalent to the Dupont law). The size 7 model is such an idempotent magma.

887 $x = y\diamond((x\diamond y)\diamond(z\diamond z))$ has spectrum $\{1,2 \mod 3\}\setminus\{7\}$. Indeed, the law characterizes exactly the semi-symmetric loops, equivalent to Mendelsohn triple systems on $M \setminus \{e\}$.

1083 $x = y \diamond ((x \diamond (y \diamond x)) \diamond y)$ has **unknown** spectrum containing 1,3,4,7,8,9. It implies that $x\Box y = x \diamond (y \diamond x)$ obeys law 14.

1719 $x = (y\diamond y)\diamond((x\diamond x)\diamond y)$ has **unknown** spectrum containing $\{0,1 \mod 3\}\cup\{5,8\}$, possibly $[1,+\infty) \setminus \{2\}$? Indeed, it is obeyed by Mendelsohn quasigroups so the spectrum contains $\{0,1 \mod 3\} \setminus \{6\}$, and magmas of size $5,6,8$ are easily found by an ATP.

2.3 Twists of the Dupont law 63

Next, we can discuss laws related to the Dupont law 63 and to law 73, its parastrophic equivalent.

63 $x = y\diamond(x\diamond(x\diamond y))$ (and 73, 118, 125, 1692) has **unknown** spectrum starting with $1,3,4,5,7,8,9,11,12,13$ and containing most sizes $n \equiv 0,1,3 \mod 4$ up to 100. Many magmas (including some of size 9 that I think are non-linear) obey $x = y \diamond (y \diamond (y \diamond x))$ in addition to law 63.

467 $x = y\diamond(x\diamond(x\diamond(y\diamond y)))$ has **unknown** spectrum starting with $1,5,7,8$. It is the Dupont law twisted by the squaring map. The spectrum contains at least all odd integers that are sums of two squares $k^2 + l^2$ (with k, l of different parities), as the quotient $\mathbb{Z}[i]/(k + il)\mathbb{Z}[i]$ equipped with the operation $x \diamond y = -\frac{k^2+l^2+1}{2}(1 + i)x + iy$ obeys this law. It has other linear models which are harder to describe. It likely has non-linear models too.

704 $x = y \diamond (y \diamond ((x \diamond x) \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is a twist of law 73 by the squaring map.

1110 $x = y \diamond ((y \diamond (x \diamond x)) \diamond y)$ has **unknown** spectrum starting with $1, 4, 5, 7, 8, 9$. It is a twist of law 73 by the squaring map.

1279 $x = y \diamond (((x \diamond x) \diamond y) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is (dual to) a twist of law 73 by the squaring map.

1516 $x = (y \diamond y) \diamond (x \diamond (x \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$. It is the Dupont law twisted by the squaring map.

2.4 Specializations of the central groupoid law 168

We know the spectrum of the central groupoid law 168 by a result of Knuth:

$$\text{spec}(\text{E168}) = \{n^2 \mid n \geq 1\}. \tag{5}$$

The spectrum of laws implied by law 168 thus contains $\{n^2 \mid n \geq 1\}$.

167 $x = (y \diamond x) \diamond (x \diamond y)$ has spectrum $\{0, 1 \mod 4\}$ see Ben Gunby-Mann's notes

1480 $x = (y \diamond x) \diamond (x \diamond (x \diamond z))$ has **conjectural** spectrum $\{1\} \cup [4, +\infty)$, checked with an ATP up to order 18. Experimentally, it is even possible to take $x \diamond y = 0$ for $\lfloor \sqrt{n} \rfloor$ different values of y (different from 0). There are models of all sizes $4 \cup [6, 11]$ obeying law 1480 and law 166 $x = (y \diamond x) \diamond (x \diamond x)$, which are much easier to find for an ATP than just law 1480.

1483 $x = (y \diamond x) \diamond (x \diamond (y \diamond z))$ has **unknown** spectrum starting with $1, 2, 4, 8, 9$.

1485 $x = (y \diamond x) \diamond (x \diamond (z \diamond y))$ has **conjectural** spectrum $\{n^2 \mid n \geq 1\} \cup \{2n^2 \mid n \geq 1\}$ as discussed at `https://leanprover.zulipchat.com/#narrow/stream/458659-Equational/topic/1485`.

1486 $x = (y \diamond x) \diamond (x \diamond (z \diamond z))$ has **unknown** spectrum containing squares and $\{n^2 + 2 \mid n \geq 3\} \cup \{13, 21\}$, discussed at `https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Understanding.20Finite.201486.20Magmas`

2.5 Isolated laws

474 $x = y \diamond (x \diamond (y \diamond (x \diamond y)))$ has spectrum $[1, +\infty) \setminus \{2, 4\}$. It has models of all odd sizes $n = 2k + 1$ (which also obey law 1685), given by (for $0, x, y$ distinct) $0 \diamond 0 = 0$, $x \diamond x = 0$, $x \diamond 0 = x$, $0 \diamond x = \sigma(x)$ an involution without fixed point, and otherwise $x \diamond y = y$. It has a model of all even sizes ≥ 6

given by an explicit multiplication table for $0 \leq x, y \leq 5$,

	0	1	2	3	4	5
0	0	2	3	4	5	1
1	1	0	5	4	3	2
2	2	3	0	1	5	4
3	3	5	4	0	2	1
4	4	2	1	5	0	3
5	5	4	3	2	1	0

(6)

together with $x \diamond x = 0$ and $x \diamond 0 = x$ for all x, and $0 \diamond x = \sigma(x)$ an involution without fixed point for $6 \leq x < n$, and $x \diamond y = y$ for all remaining entries, namely $\min(x, y) \geq 1$ and $\max(x, y) \geq 6$ and $x \neq y$.

670 $x = y \diamond (x \diamond ((x \diamond y) \diamond y))$ has **unknown** spectrum starting with $1, 4, 5$, not 6 nor 7

677 $x = y \diamond (x \diamond ((y \diamond x) \diamond y))$ has **unknown** spectrum. It is the source of the famous last surviving implication.

907 $x = y \diamond ((y \diamond x) \diamond (x \diamond y))$ has **unknown** spectrum containing $1, 3, 7, 9, 13$, and not $2, 4, 5, 6$. Models tend to either have a left identity or to be commutative and idempotent.

1076 $x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$ has **unknown** spectrum starting with $1, 5$, and not having 6 nor 7.

1286 $x = y \diamond (((x \diamond y) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 7$.

1313 $x = y \diamond (((y \diamond x) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7$.

1489 $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ has **conjectural** spectrum $[1, +\infty) \setminus \{2, 4\}$, see below.

1685 $x = (y \diamond x) \diamond ((x \diamond y) \diamond y)$ has spectrum $[1, +\infty) \setminus \{2\}$. It has models of all odd sizes $n = 2k + 1$ (which also obey law 474), given by (for $0, x, y$ distinct) $0 \diamond 0 = 0$, $x \diamond x = 0$, $x \diamond 0 = x$, $0 \diamond x = \sigma(x)$ an involution without fixed point, and otherwise $x \diamond y = y$. For even sizes $n = 2k + 4 \geq 4$, take $x \diamond y$ given by the following table

	0	1	2	3
0	0	1	2	3
1	3	2	1	0
2	1	0	3	2
3	2	3	0	1

(7)

together with $x \diamond 0 = x$ and $x \diamond y = (y \diamond 0) \diamond 0$ for $1 \leq y \leq x$, and $x \diamond x = 0$ and $0 \diamond x = \sigma(x)$ for $4 \leq x$ (with σ an involution of $[4, n-1]$ without fixed points), and all other entries $x \diamond y = y$ (namely for $y \geq 4$ and $x \neq 0, y$).

2.5.1 Exploration of the equation 1489

Equation 1489 $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ has **conjectural** spectrum $[1, +\infty) \setminus \{2, 4\}$, checked up to order 21 included. **Curiously**, the smallest commutative models of law 1489 have size $1, 5, 21$ —we do not investigate further.

For all sizes except 5, there appears to be models obeying 1489 and law 4321 $x \diamond (y \diamond x) = y \diamond (x \diamond y)$. Together, these laws imply law 3 $x = x \diamond x$ and law 28 $x = (y \diamond x) \diamond x$ and law 1481 $x = (y \diamond x) \diamond (x \diamond (y \diamond x))$. In these models $x \Box y = x \diamond (y \diamond x)$ is commutative and obeys law 323 $x \Box y = x \Box (x \Box y)$, or equivalently law 333 $x \Box y = y \Box (x \Box y)$. One can define a graph with edges

$$(x \to y) \Leftrightarrow (x \Box y = y) \Leftrightarrow (y \Box x = y) \Leftrightarrow (\exists z, y = x \Box z) \Leftrightarrow (\exists z, y = z \Box x) \Leftrightarrow (x \diamond y = y). \tag{8}$$

Experimentally, the in and out degrees of each node are equal. Experimentally that degree (not counting self-loops) is between 3 and $\lfloor (n-1)/2 \rfloor$ included. For sizes $[3, 6, 7, 8, 9]$ at least, the graph can have cyclic symmetry, with degrees $[1, 2, 3, 3, 3]$, respectively. (However, the operation $\diamond$ itself, or even $\Box$, does not have cyclic symmetry, only approximate symmetry.) An example model on $[0, 8]$ whose graph is cyclically symmetric has the operation $\diamond$ given by

[0, 3, 8, 3, $\underline{6}$, $\overline{2}$, 7, 7, 8],
[0, 1, 4, 0, 4, $\underline{7}$, $\underline{7}$, 8, 8],
[0, 1, 2, 5, 1, 5, $\overline{7}$, $\underline{8}$, 0],
[1, 1, 2, 3, 6, 2, 6, $\overline{8}$, $\underline{0}$],
[$\underline{1}$, 2, 2, 3, 4, 7, 3, 7, $\overline{0}$],
[$\overline{1}$, $\underline{2}$, 3, 3, 4, 5, 8, 4, 8],
[0, $\underline{3}$, $\overline{8}$, 4, 4, 5, 6, 0, 5],
[6, 1, $\underline{4}$, $\overline{0}$, 5, 5, 6, 7, 1],
[2, 7, 2, $\underline{5}$, $\overline{1}$, 6, 6, 7, 8]

and the operation $\Box$ given by

```
[[0, 0, 0, 3, 3, 3, 0, 7, 8],
 [0, 1, 1, 1, 4, 4, 0, 1, 8],
 [0, 1, 2, 2, 2, 5, 0, 1, 2],
 [3, 1, 2, 3, 3, 3, 6, 1, 2],
 [3, 4, 2, 3, 4, 4, 4, 7, 2],
 [3, 4, 5, 3, 4, 5, 5, 5, 8],
 [0, 0, 0, 6, 4, 5, 6, 6, 6],
 [7, 1, 1, 1, 7, 5, 6, 7, 7],
 [8, 8, 2, 2, 2, 8, 6, 7, 8]]
```

An example model on $[0, 16]$ without symmetry is given by

```
[0, 1, 1, 1, 4, 4, 3, 9, 1, 9,10,10, 2,13,13, 8,14,
 2, 1, 2, 3, 3, 2, 7, 7, 8, 0, 8, 2,15,12,11,15, 5,
```

```
 0, 0, 2, 1, 4, 5, 4, 5, 1,10,10,11,12,12, 8,11,14,
 5, 4, 6, 3, 4, 2, 6, 7, 7, 6, 0,11,11,13,11, 8,13,
 5, 1, 6, 1, 4, 5, 6, 1, 1, 9,10, 9,10,10,15,16,12,
 0, 6, 7, 8, 0, 5, 3, 7, 9, 9, 0,11, 6,11,10,16,16,
 7, 1, 2, 9, 2, 8, 6, 1, 1, 9,10, 2,12,10,15,15,12,
 0, 6, 2, 8, 2, 2, 6, 7, 8, 0, 0,11,14,10,11,11,12,
11,10, 9, 3,12, 5,13, 3, 8, 5,10, 2,14,15,14,15, 5,
 7, 6, 2, 3,11, 8, 3, 7, 8, 9, 2,11,11,12,11,11,13,
11, 1, 9,12,12,13,13,14, 1, 9,10,11,12,13,14, 1,14,
 0,12,15,12, 4,13,14,14,13, 4, 0,11,12,13,14,15, 5,
14, 1,13, 3, 4,15,16, 3, 8, 6, 4, 3,12,13, 8, 1,16,
14, 0, 2,16, 2, 5, 6, 1, 8,10, 6, 5, 2,13,14, 8,16,
 0, 6,13, 8, 2,15, 6, 7,12, 0,16, 7,12, 0,14, 6,16,
14,12, 2,16, 0, 5,14, 5,13,10, 4, 2,12,13,14,15, 5,
11,12, 9, 3, 2,15, 6, 1,13, 6,10, 2, 6, 3,10,15,16 ]
```

and the operation $x\square y$ is given by

```
[0, 1, 0, 4, 4, 0, 9, 0,10, 9,10, 0,13,13, 0,13,10,
 1, 1, 2, 3, 1, 7, 1, 7, 8, 7, 1,15, 1, 2, 7,15,15,
 0, 2, 2, 4, 4, 5, 2, 2,10, 2,10,11,12, 2,12, 2,10,
 4, 3, 4, 3, 4, 7, 6, 7, 3, 3,11,11, 3,13, 7,13, 3,
 4, 1, 4, 4, 4, 5, 6, 6,10, 9,10, 4, 4, 6, 6, 5, 6,
 0, 7, 5, 7, 5, 5, 9, 7, 5, 9,11,11,16, 5,16, 5,16,
 9, 1, 2, 6, 6, 9, 6, 6,10, 9,10,15,12, 6, 6,15, 6,
 0, 7, 2, 7, 6, 7, 6, 7, 8, 7,11,11, 8, 6, 7, 2, 6,
10, 8,10, 3,10, 5,10, 8, 8, 8,10,15, 8, 8,14,15,15,
 9, 7, 2, 3, 9, 9, 9, 7, 8, 9, 9,11, 3, 2, 7, 2, 3,
10, 1,10,11,10,11,10,11,10, 9,10,11,12,13,14,12,10,
 0,15,11,11, 4,11,15,11,15,11,11,11,12,13,14,15,15,
13, 1,12, 3, 4,16,12, 8, 8, 3,12,12,12,13,12,12,16,
13, 2, 2,13, 6, 5, 6, 6, 8, 2,13,13,13,13,14,13,16,
 0, 7,12, 7, 6,16, 6, 7,14, 7,14,14,12,14,14,14,16,
13,15, 2,13, 5, 5,15, 2,15, 2,12,15,12,13,14,15,15,
10,15,10, 3, 6,16, 6, 6,15, 3,10,15,16,16,16,15,16]
```

and the (in or out) degrees are $[5, 5, 5, 5, 6, 6, 6, 5, 5, 5, 5, 5, 6, 6, 6, 4, 5]$ so this probably not the most symmetrical model possible.

3 Simple and (sub-)directly irreducible spectrum (todo)

We explain here in a streamlined way the results of the investigation discussed at https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Simple.20and.20.28sub.29directly.20irreducible.20spectrum/near/493052424

Table 3: Number of magmas satisfying various laws up to order 2, sorted by their large-n behaviour. Law 2 is not included here.

Law A	Equation	$\log N_A(n)$	$\log I_A(n)$	"Phase"
Law 1	$x = x$	$n^2 \log n + O(n \log n)$		vacuum
Law 3	$x = x \diamond x$	$n^2 \log n + O(n \log n)$		vacuum
Law 8	$x = x \diamond (x \diamond x)$	$n^2 \log n + O(n \log n)$		vacuum
Law 11	$x = x \diamond (y \diamond y)$	$n^2 \log n + O(n \log n)$		vacuum
Law 40	$x \diamond x = y \diamond y$	$n^2 \log n + O(n \log n)$		vacuum
Law 9	$x = x \diamond (x \diamond y)$	$n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 10	$x = x \diamond (y \diamond x)$	$n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 16	$x = y \diamond (y \diamond x)$	$(1/2)n^2 \log n + O(n^2)$		gas ($p = 1/2$)
Law 43	$x \diamond y = y \diamond x$	$(1/2)n^2 \log n + O(n \log n)$		gas ($p = 1/2$)
Law 14	$x = y \diamond (x \diamond y)$	**likely** $(1/3)n^2 \log n + O(n^2)$		gas ($p = 2/3$)
Law 38	$x \diamond x = x \diamond y$	$n \log n$	$\sim (1.08\ldots)n$	solid
Law 4	$x = x \diamond y$	0	0	crystal
Law 13	$x = y \diamond (x \diamond x)$	$\sim (1/2)n \log n$	$\sim \log n$	crystal
Law 41	$x \diamond x = y \diamond z$	$\log n$	0	crystal

4 Counting magmas satisfying a law

Ben Gunby-Mann started investigating the number of magmas satisfying low-lying laws in https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Equations.20with.20full.20spectrum/near/490261916 Continuing this investigation leads to asymptotic counts for most laws up to order 3. There remains some challenges for

- the semi-symmetric law 14 and its twist law 66 where the lower bound is only proven for some families of sizes;
- for law 65, which only has a lower bound;
- for law 63 (and 73) where even the spectrum remains unknown.
- Laws of many other shapes!

4.1 Order up to 2

We determine here the number of magmas on a carrier set S of cardinal n subject to an equational law A. We consider two related numbers:

- $N_A(n)$ counting labeled magmas (namely counting magma operations on a fixed set $\{1, \ldots, n\}$), and
- $I_A(n)$ counting isomorphism classes of magmas.

There are more general notions of equivalence (parastrophy etc.), but we do not consider them. For many laws A one has $N_A(n) \sim n! I_A(n)$ because a typical magma has no symmetry. It is often easier to determine (or estimate) $N_A(n)$ and use the following obvious inequalities to control the asymptotics of $I_A(n)$,

$$\frac{1}{n!} N_A(n) \le I_A(n) \le N_A(n). \tag{9}$$

We summarize results in Table 4.

In this discussion we consider in turn one law in each equivalence class, omitting laws whose dual is (equivalent to) a lower-numbered law. Note that if law A implies law B then $N_B(n) \le N_A(n)$ and $I_B(n) \le I_A(n)$. In particular, law 3 (which has a lot of models) implies all single-variable laws.

- Law 1 ($x = x$): the magma operation is unconstrained and consists of choosing n^2 times among n choices so $N_1(n) = n^{n^2}$.
- Law 2 ($x = y$): only possible for $n = 1$, so $N_2(n) = I_2(n) = \delta_{n=1}$.
- Law 3 ($x = x \diamond x$): diagonal entries of the Cayley table are fixed, the others are arbitrary so $N_3(n) = n^{n^2-n}$.
- Law 4 ($x = x \diamond y$): the law fully fixes the operation so $N_4(n) = I_4(n) = 1$.
- Law 8 ($x = x \diamond (x \diamond x)$): each left-multiplication is constrained independently, either $L_x(x) = x$ and the $n-1$ remaining $L_x(y)$ are arbitrary, or L_x maps x to another value itself mapped to x. Thus $N_8(n) = ((2n-1)n^{n-2})^n > N_3(n)$.
- Law 9 ($x = x \diamond (x \diamond y)$): each left-multiplication L_x is constrained independently, and this law implies idempotence. The choice of L_x amounts to the choice of subset $L_x^{-1}(x) \setminus x$ and of map $(M \setminus L_x^{-1}(x)) \to L_x^{-1}(x) \setminus x$, so

$$N_9(n) = \Big(\sum_{k=1}^{n-1} \binom{n-1}{k} k^{n-1-k}\Big)^n = (\text{A000248}(n-1))^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \tag{10}$$

- Law 10 ($x = x \diamond (y \diamond x)$): such a magma is characterized by an unoriented graph (which can have loops) and for every $x \in M$ a map from non-neighbors of x to neighbors of x, so there are (denoting by $\deg_E(i)$ the number of edges $(i, \dots)$ or $(\dots, i)$ in E)

$$\sum_{E \subset \{(i,j), 1 \le i \le j \le n\}} \prod_{i=1}^{n} \big(\deg_E(i)\big)^{n - \deg_E(i)}. \tag{11}$$

Consider a graph with $\deg_E(i) \simeq n/\log n$ for all i. The corresponding term scales as $(n/\log n)^{n^2(1-1/\log n)} = e^{n^2(\log n - \log\log n + O(1))}$, which is thus a lower bound for $N_{10}(n)$. To get an upper bound, we consider the function $f(x) = \log(x^{n-x}) = (n-x)\log x$ for $x > 0$. One has $f'(x) = -\log x - 1 + n/x$ and $f''(x) = -1/x - n/x^2 < 0$ so it is concave, with a maximum at $x \log x + x = n$, namely (for large n) $x = n(\log n)^{-1} + O(n(\log n)^{-2} \log\log n)$, from which we find $f(x) \leq n\log n - n\log\log n + O(n)$. Every term in the sum is thus bounded by $e^{n^2(\log n - \log\log n + O(1))}$. On the other hand there are $2^{n(n+1)/2} = e^{O(n^2)}$ graphs, which leads to the asymptotics $N_{10}(n) = e^{n^2(\log n - \log\log n + O(1))}$.

- Law 11 ($x = x \diamond (y \diamond y)$): such magmas are characterized by the set Q of squares, a squaring map $M \setminus Q \to Q$ (for $x \in Q$, $x \diamond x = x$), and the remaining entries $x \diamond y$ for $y \in M \setminus Q$ and $x \in M \setminus \{y\}$, so

$$N_{11}(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k} n^{(n-1)(n-k)} = n^{n^2-2n+O(1)}, \tag{12}$$

  with most of these being accounted for by unipotent magmas (all squares equal, $k = 1$).

- Law 13 ($x = y \diamond (x \diamond x)$): the operation is right-projection composed with an involution, so

$$N_{13}(n) = \text{A000085}(n) = e^{(1/2)n(\log n - 1) + \sqrt{n} + O(1)}. \tag{13}$$

  Modulo isomorphism, only the number of 2-cycles matters, which ranges from 0 to $\lfloor n/2 \rfloor$, so $I_{13}(n) = \lfloor n/2 \rfloor + 1$.

- Law 14 ($x = y \diamond (x \diamond y)$): $N_{14}(n) = \text{A076016}(n)$ and $I_{14}(n) = \text{A076017}(n)$. The asymptotics are difficult to establish so we return to it momentarily.

- Law 16 ($x = y \diamond (y \diamond x)$): each left-multiplication is constrained independently to be an involution, so $N_{16}(n) = N_{13}(n)^n = \text{A000085}(n)^n$. See (13) for asymptotics.

- Law 38 ($x \diamond x = x \diamond y$): the operation is left-projection composed with squaring, which is an arbitrary endofunction, so $N_{38}(n) = n^n$, and $I_{38}(n) = A001372(n) \sim C_2 C_1^n/\sqrt{n}$ for some constants $C_1 \simeq 2.96$ and $C_2 \simeq 0.443$, see `https://oeis.org/A001372`.

- Law 40 ($x \diamond x = y \diamond y$): magmas in which all elements square to the same element (which we have to pick) so $N_{40}(n) = n^{n^2-n+1}$.

- Law 41 ($x \diamond x = y \diamond z$): all products are equal to the same element, so $N_{41}(n) = n$, and all choices are isomorphic so $I_{41}(n) = 1$.

- Law 43 ($x \diamond y = y \diamond x$): commutativity means we only pick $n(n+1)/2$ entries in the Cayley table, so $N_{43}(n) = n^{n(n+1)/2}$.

4.2 Interlude: semi-symmetric quasigroups

Our aim is to show that $\log N_{14}(n) \sim (1/3)n^2 \log n$. We first determine an upper bound. The magma operations can be enumerated by the following procedure. First we have to choose the squaring map $S\colon M \to M$. Then, consider the list L of pairs (x,y) with $x \neq y$ and $x \neq S(y)$ and $y \neq S(x)$, and sort L in lexicographic order (say). For the first pair in that list L, decide the product $x \diamond y \in M \setminus \{x,y\}$; it must be such that the pairs $(y, x \diamond y)$ and $(x \diamond y, x)$ are in the list L. Delete these two pairs from the list L. Then move on to choosing the product for the next pair in L, and so on. This process either exhausts L or gets stuck at a point where there is no valid choice of $x \diamond y$. Every step reduces $|L|$ by 3, so the process stops in at most $(n^2-n)/3$ steps, and we get

$$N_{14}(n) \leq n^n (n-2)^{(n^2-n)/3} = e^{(1/3)n^2 \log n + O(n^2)}. \tag{14}$$

Known numerical values of A076016(n) are not sufficient to test whether this coefficient is indeed correct.

We show that this asymptotics is **correct for** $n = 2^k m$ **with** $k \to +\infty$ **and fixed** $m \geq 1$, as follows.

It is known that the number $N_{\mathrm{Latin}}(n) =$ A002860(n) of Latin squares (labelled quasigroups) is bounded below as

$$N_{\mathrm{Latin}}(n) \geq \left(n!^2/n^n\right)^n = e^{n^2(\log n - 2) + O(n \log n)}. \tag{15}$$

(An intuition for this result is that a random map $[0, n-1) \to [0, n-1)$ has probability roughly e^{-n} to be a permutation (since $n!/n^n = e^{-n+O(\log n)}$) and we want $2n$ such conditions, for each left and right multiplications, which explains the e^{-2n^2} factor.)

proof sketch at https://math.stackexchange.com/questions/4166583/explanation-for-the-fracn2nnn2-lower-bound-on-the-number-of-lat

Given a quasigroup $(Q, \cdot)$ of cardinality $n/2$ and a semi-symmetric quasigroup with the same underlying set $(Q, \circ)$ (namely $\circ$ obeys law 14), we define $M = Q \times \{0,1\}$ with the operation

$$\begin{aligned} (x,0) \diamond (y,0) &= (x \cdot y, 1), \\ (y,0) \diamond (x \cdot y, 1) &= (x, 0), \\ (x \cdot y, 1) \diamond (x, 0) &= (y, 0), \\ (x,1) \diamond (y,1) &= (x \circ y, 1). \end{aligned} \tag{16}$$

The second and third identities here define fully the products $(\_,0) \diamond (\_,1)$ and $(\_,1) \diamond (\_,0)$ because $(Q, \cdot)$ is a quasigroup. This operation obeys law 14, thus

$$N_{14}(n) \geq N_{14}(n/2) N_{\mathrm{Latin}}(n/2) = N_{14}(n/2) e^{(1/4)n^2(\log n - 2 - \log 2) + O(n \log n)}. \tag{17}$$

Iterating k times this identity (for $n = 2^k m$), and using $N_{14}(m) \geq 1$ (thanks to the explicit model $\mathbb{Z}/m\mathbb{Z}$ with $x \circ y = -x - y \mod m$),

$$\log N_{14}(n) \geq \frac{1 - 2^{-2v_2(n)}}{3} n^2 \log n - \left(\frac{2}{3} + \frac{4}{9}\log 2\right) n^2 + O(n (\log n)^2), \tag{18}$$

Table 4: Number of magmas satisfying various laws of shape $\_ = \_\diamond(\_\diamond(\_\diamond\_))$, sorted by their large-$n$ behaviour.

Law A	Equation	$\log N_A(n)$	$\log I_A(n)$	"Phase"
Law 47	$x = x \diamond (x \diamond (x \diamond x))$	$n^2 \log n + O(n \log n)$		vacuum
Law 50	$x = x \diamond (x \diamond (y \diamond y))$	$n^2 \log n + O(n \log n)$		vacuum
Law 56	$x = x \diamond (y \diamond (y \diamond y))$	$n^2 \log n + O(n \log n)$		vacuum
Law 48	$x = x \diamond (x \diamond (x \diamond y))$	$n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 49	$x = x \diamond (x \diamond (y \diamond x))$	$\geq n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 52	$x = x \diamond (y \diamond (x \diamond x))$	$\geq n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 53	$x = x \diamond (y \diamond (x \diamond y))$	$\geq n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 55	$x = x \diamond (y \diamond (y \diamond x))$	$\geq n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 58	$x = x \diamond (y \diamond (z \diamond x))$	$\geq n^2(\log n - \log\log n) + O(n^2)$		gas ($p = 0$)
Law 75	$x = y \diamond (y \diamond (y \diamond x))$	$(2/3)n^2 \log n + O(n^2)$		gas ($p = 1/3$)
Law 65	$x = y \diamond (x \diamond (y \diamond x))$	$\geq (1/2)n^2 \log n + O(n^2 \log\log n)$		gas ($\boldsymbol{p \leq 1/2}$)
Law 72	$x = y \diamond (y \diamond (x \diamond x))$	$(1/2)n^2 \log n + O(n^2)$		gas ($p = 1/2$)
Law 66	$x = y \diamond (x \diamond (y \diamond y))$	**likely** $(1/3)n^2 \log n + O(n^2)$		gas ($p = 2/3$)
Law 63	$x = y \diamond (x \diamond (x \diamond y))$	**unknown spectrum**		
Law 73	$x = y \diamond (y \diamond (x \diamond y))$	same as law 63		
Law 62	$x = y \diamond (x \diamond (x \diamond x))$	$\sim (2/3)n \log n$	$\sim \log n$	crystal

where $k = v_2(n)$ is the 2-adic valuation of n. For n in the sequence $\{2^k m \mid k \geq 1\}$, the first term behaves as $(1/3)n^2 \log n + O(\log n)$. In fact, **if one could prove that $N_{14}(n)$ is monotonically increasing,** then the information we have would be enough to deduce $\log N_{14}(n) \sim (1/3)n^2 \log n$. Alternatively, one could try to **tweak the construction above** to allow for an additional element: instead of extending a semi-symmetric quasigroup by a quasigroup, we could maybe extend a semi-symmetric loop by a loop and decide whether to identify the two identity elements or not depending on parity of n?

NB: Law 887 characterizes semi-symmetric loops, namely magmas satisfying both law 14 and the unipotence law 40 $x \diamond x = y \diamond y$ (the common value of the squares is then a two-sided identity e). Such magmas are in one-to-one correspondence with Mendelsohn triple systems on the underlying set with e removed. Likewise, law 4961 characterizes idempotent semi-symmetric quasigroups, namely magmas satisfying both law 14 and the idempotence law 3 $x \diamond x = x$, which are in one-to-one correspondence with Mendelsohn triple systems on the set itself. We have $I_{887}(m) = I_{4961}(m-1) = \text{A076021}(m-1)$.

4.3 Order 3: first shape $\_\_ = \_\_ \diamond (\_\_ \diamond (\_\_ \diamond \_\_))$

- Law 47 ($x = x \diamond (x \diamond (x \diamond x))$) is a single-variable law hence $N_{47} \geq N_3$. In detail, each left multiplication is chosen separately, either $x \diamond x = x$ and other entries are unspecified, or one has to choose distinct $x \diamond x, x \diamond (x \diamond x)$, so $N_{47}(n) = ((2n^2 - 3n + 2)n^{n-3})^n$.

- Law 48 ($x = x \diamond (x \diamond (x \diamond y))$) is similar to law 9; it constrains each left multiplication L_x independently to cube to the constant x (and in particular $L_x(x) = x$). Such maps are in one-to-one correspondence with maps $f\colon M \setminus \{x\} \to M \setminus \{x\}$ with $f(f(y)) = f(f(f(y)))$, by assigning $L_x(y) = f(y)$ if that is different from y and otherwise $L_x(y) = x$. Thus $N_{48}(n) = \text{A000949}(n-1)^n$, whose asymptotics are known.

- Law 49 ($x = x \diamond (x \diamond (y \diamond x))$) allows a class of magmas with $x \diamond y = x$ for $\min(x, y) < n/\log n$ and $x \diamond y < n/\log n$ for $\min(x, y) \geq n/\log n$, of which there are $e^{n^2(\log n - \log\log n + O(1))}$. This is **probably** also the correct asymptotics, but one would have to prove an upper bound. We dub it "gas ($p = 0$)" even though it could be closer to a "vacuum".

- Law 50 ($x = x \diamond (x \diamond (y \diamond y))$) is a consequence of law 9, so $N_{50}(n) \geq e^{n^2(\log n - \log\log n + O(1))}$. We can do better. On $\{1, \dots, n\}$ with $n \geq 3$, set all $x \diamond x = 1$ so that the law reduces to a condition $L_x(L_x(1)) = x$ for all $x \neq 1$. This is easily achieved by taking $L_x(1) \in M \setminus \{1, x\}$ and then L_x of that to be x. The remaining $n - 3$ entries of L_x are arbitrary. Thus

$$N_{50}(n) \geq (n-2)^n n^{n(n-3)} = e^{n^2 \log n + O(n \log n)}. \tag{19}$$

- Law 52 ($x = x \diamond (y \diamond (x \diamond x))$): the magmas constructed for law 10, restricted to idempotent magmas, show that $N_{52}(n) \geq e^{n^2(\log n - \log\log n + O(1))}$. The reverse inequality is unclear, same comments as for law 49 apply.

- Law 53 ($x = x \diamond (y \diamond (x \diamond y))$): one can construct many magmas of size n as follows. Pick an idempotent semi-symmetric quasigroup $(Q, \cdot)$ (namely a magma satisfying laws 3 and 14, such as $\mathbb{Z}/3\mathbb{Z}$ equipped with $x \cdot y = -x - y$), and a surjective map $\pi\colon M \to Q$ and take

$$\begin{aligned} &x \diamond y = x && \text{if } \pi(x) = \pi(y), \\ &x \diamond y \in \pi^{-1}(\pi(x) \cdot \pi(y)) && \text{otherwise.} \end{aligned} \tag{20}$$

  Note that $\pi(x \diamond y) = \pi(x) \cdot \pi(y)$ in all cases since $(Q, \cdot)$ is idempotent. Since $\pi(y \diamond (x \diamond y)) = \pi(y) \cdot (\pi(x) \cdot \pi(y)) = \pi(x)$ the first rule applies when computing $x \diamond (y \diamond (x \diamond y))$, which is thus equal to x as wanted. This construction is very flexible. Denoting $q = |Q|$, there are at least $\lfloor n/q \rfloor^{q(q-1)\lfloor n/q \rfloor^2}$ such magmas. At large n, taking $q = \log n$, we obtain $e^{n^2(\log n - \log\log n) + O(n^2)}$ magmas.

- Law 55 ($x = x \diamond (y \diamond (y \diamond x))$): implied by law 58, below, hence $N_{55} \geq N_{58}$.

- Law 58 ($x = x \diamond (y \diamond (z \diamond x))$): for some surjection $\pi\colon M \to Q$ for some set Q, consider

$$\begin{aligned} &x \diamond y = x && \text{if } \pi(x) = \pi(y), \\ &x \diamond y \in \pi^{-1}(y) && \text{if } \pi(x) \neq \pi(y). \end{aligned} \tag{21}$$

In all cases $\pi(x \diamond y) = \pi(y)$, so $\pi(y \diamond (z \diamond x)) = \pi(x)$, so when computing $x \diamond (y \diamond (z \diamond x))$ the first case applies and we get x as desired. For a fixed set Q of cardinality q, we can take the sets $\pi^{-1}(i)$ to each have at least $\lfloor n/q \rfloor$ elements, and we find $N_{58}(n) \geq \lfloor n/q \rfloor^{q(q-1)\lfloor n/q \rfloor^2}$. Taking $q = \log n$ as for law 53 we get the announced asymptotics.

- Law 56 ($x = x\diamond(y\diamond(y\diamond y))$): a class of models is given by $0\diamond 0 = 0$, $1\diamond 0 = 1$, $1 \diamond 1 = 2$, $1 \diamond 2 = 0$, and, for $x \geq 2$, $x \diamond 0 = x$, $x \diamond 1 = 0$ and $x \diamond x = 1$. Indeed, these identities imply $y \diamond (y \diamond y) = 0$ and $x \diamond 0 = x$ for all x, y. There remains $n(n-2)$ entries to specify arbitrarily, so $N_{56}(n) \geq n^{n(n-2)}$.

- Law 62 ($x = y \diamond (x \diamond (x \diamond x))$): the magma operation is right-projection composed with a permutation σ with $\sigma^{\circ 3} = \mathrm{id}$, so $N_{62}(n) = \text{A001470}(n)$. Up to isomorphism, only the number of 3-cycles matters, so $I_{62}(n) = \lfloor n/3 \rfloor + 1$.

- Law 65 ($x = y \diamond (x \diamond (y \diamond x))$): a class of such magmas is build as follows. Fix a surjection $\pi \colon M \to Q$ to some set. For each $x \in M$ and $i \in Q$ with $\pi(x) \neq i$, select an involution $f_{x,i} \colon \pi^{-1}(i) \to \pi^{-1}(i)$, and for $\pi(x) = i$ set $f_{x,i}$ to be the identity. Then define $x \diamond y = f_{x,\pi(y)}(y)$. By construction $\pi(x \diamond y) = \pi(y)$ so x and $y \diamond x$ have the same image under π and hence $x \diamond (y \diamond x) = y \diamond x$. Thus, $y \diamond (x \diamond (y \diamond x)) = y \diamond (y \diamond x) = f^{\circ 2}_{y,\pi(x)}(x) = x$. The number of magmas built this way, for a fixed q and surjection whose preimages each have at least $\lfloor n/q \rfloor$ elements, is a power of the number N_{13} of involutions,

$$N_{65}(n) \geq \big(N_{13}(\lfloor n/q \rfloor)\big)^{n(q-1)} \overset{q=\log n}{=} e^{(1/2)n^2(\log n - \log\log n) + O(n^2)}. \tag{22}$$

- Law 66 ($x = y \diamond (x \diamond (y \diamond y))$): such magmas are in one-to-one correspondence with Mendelsohn quasigroups (law 4961, equivalently idempotence law 3 and semi-symmetry law 14) equipped with an involutive automorphism. The construction of law 14 magmas **can probably** be restricted to idempotent magmas without changing the leading asymptotics. The involutive automorphism changes the number of such magmas by a factor between 1 (it can be the identity) and $n!$ (it has to be a bijection), which also leaves the leading asymptotics unchanged.

- Law 72 ($x = y \diamond (y \diamond (x \diamond x))$): squaring is a permutation σ, and all left-multiplications L_y square to σ^{-1}. In the special case $\sigma = \mathrm{id}$ left-multiplications L_y are simply involutions of $M \setminus \{y\}$. Thus, $N_{72}(n) \geq \text{A000085}(n-1)^n = e^{(1/2)n^2 \log n + O(n^2)}$, see (13) for precise asymptotics. To find an upper bound, consider that for any choice of permutation σ (of which there are $e^{O(n \log n)}$) one can choose each L_y one entry at a time; each time we choose one entry another entry is automatically fixed, so that we have at most $n(n-2)(n-4)\dots$ choices of L_y.

- Law 75 ($x = y \diamond (y \diamond (y \diamond x))$): each left multiplication is chosen independently to be a permutation of order 1 or 3, so one has $N_{75}(n) =$ A001470$(n)^n = e^{(2/3)n^2(\log n - 1) + n^{4/3} + O(n \log n)}$.

4.4 Order 3: second shape $\_ = \_ \diamond ((\_ \diamond \_) \diamond \_)$

4.5 Order 3: third shape $\_ = (\_ \diamond \_) \diamond (\_ \diamond \_)$

4.6 Order 3: fourth shape $\_ \diamond \_ = \_ \diamond (\_ \diamond \_)$

4.7 Selected higher-order laws

The following have not been verified as thoroughly.

- Law 411 ($x = x \diamond (x \diamond (x \diamond (x \diamond x)))$)
- Law 412 ($x = x \diamond (x \diamond (x \diamond (x \diamond y)))$): one has $N_{412}(n) =$ A000950$(n)^n$.
- Law 513 ($x = y \diamond (y \diamond (y \diamond (y \diamond x)))$): one has $N_{513}(n) =$ A001472$(n)^n = e^{(3/4)n^2(\log n - 1) + n^{3/2}/2 + n^{5/4} + O(nlogn)}$.
- Law 3306 ($x \diamond y = x \diamond (x \diamond (x \diamond y))$): one has $N_{3306}(n) =$ A060905$(n)^n$.
- Law 3254 ($x \diamond x = x \diamond (x \diamond (x \diamond y))$)
- Law 4268 ($x \diamond (x \diamond x) = x \diamond (x \diamond y)$): left-multiplications L_x square to constant maps, almost like law 9 except that the constant also has to be chosen, so

$$N_{4268}(n) = n^n N_9(n) = (n\text{A000248}(n-1))^n = e^{n^2(\log n - \log\log n) + O(n^2)}. \quad (23)$$