for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and some real-valued function σ on U. In particular, the characteristic function e^{σ} is related to f by $e^{\sigma(\mathbf{x})} = |df_{\mathbf{x}}\mathbf{v}|/|\mathbf{v}|$ for all $\mathbf{v} \neq 0$. We begin with a difficult lemma giving a strong implication of the conformality of f on the function σ. Throughout the remainder of this chapter we will assume that f is of class C^4 and that the dimension $n \geq 3$.

Lemma 5.1. *Let f be a conformal map with characteristic function $e^{\sigma(\mathbf{x})}$. If σ is not a constant, then $d^2(e^{-\sigma})_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \alpha\langle \mathbf{v}, \mathbf{w}\rangle$ for some constant α.*

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{\cancel{n}3}$ be ~~n mutually orthogonal unit~~ three vectors in $\mathbb{R}^n$. Then $\langle df_{\mathbf{x}}\mathbf{v}_i, df_{\mathbf{x}}\mathbf{v}_j\rangle = e^{2\sigma}\langle \mathbf{v}_i, \mathbf{v}_j\rangle$. Differentiating this, we have (see, e.g., Section 3.2)

$$\langle d^2 f_{\mathbf{x}}(\mathbf{v}_i, \mathbf{v}_k), df_{\mathbf{x}}(\mathbf{v}_j)\rangle + \langle df_{\mathbf{x}}(\mathbf{v}_i), d^2 f_{\mathbf{x}}(\mathbf{v}_j, \mathbf{v}_k)\rangle$$
$$= 2e^{2\sigma}\langle \mathbf{v}_i, \mathbf{v}_j\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_k).$$

Cyclicly permuting the indices i, j, k, we also have

$$\langle d^2 f_{\mathbf{x}}(\mathbf{v}_k, \mathbf{v}_j), df_{\mathbf{x}}(\mathbf{v}_i)\rangle + \langle df_{\mathbf{x}}(\mathbf{v}_k), d^2 f_{\mathbf{x}}(\mathbf{v}_i, \mathbf{v}_j)\rangle$$
$$= 2e^{2\sigma}\langle \mathbf{v}_k, \mathbf{v}_i\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_j),$$
$$\langle d^2 f_{\mathbf{x}}(\mathbf{v}_j, \mathbf{v}_i), df_{\mathbf{x}}(\mathbf{v}_k)\rangle + \langle df_{\mathbf{x}}(\mathbf{v}_j), d^2 f_{\mathbf{x}}(\mathbf{v}_k, \mathbf{v}_i)\rangle$$
$$= 2e^{2\sigma}\langle \mathbf{v}_j, \mathbf{v}_k\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_i).$$

Adding the last two equations and subtracting the first gives

$$\langle d^2 f_{\mathbf{x}}(\mathbf{v}_j, \mathbf{v}_i), df_{\mathbf{x}}(\mathbf{v}_k)\rangle$$
$$= e^{2\sigma}\left(\langle \mathbf{v}_k, \mathbf{v}_i\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_j) + \langle \mathbf{v}_j, \mathbf{v}_k\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_i) - \langle \mathbf{v}_i, \mathbf{v}_j\rangle d\sigma_{\mathbf{x}}(\mathbf{v}_k)\right).$$

Therefore, since $\{df_{\mathbf{x}}(\cancel{\mathbf{v}_k})\}$ is a ~~n orthogonal basis~~ surjective endomorphism of $\mathbb{R}^n$, it follows that ~~$\{e^{-\sigma}df_{\mathbf{x}}(\mathbf{v}_k)\}$ is an orthonormal basis, and~~ for ~~$i \neq j$~~ $v_1 \perp v_2$ we have

$$(5.1) \qquad d^2 f_{\mathbf{x}}(\mathbf{v}_{\cancel{j}2}, \mathbf{v}_{\cancel{i}1}) = d\sigma_{\mathbf{x}}(\mathbf{v}_{\cancel{j}2})df_{\mathbf{x}}(\mathbf{v}_{\cancel{i}1}) + d\sigma_{\mathbf{x}}(\mathbf{v}_{\cancel{i}1})df_{\mathbf{x}}(\mathbf{v}_{\cancel{j}2}),$$

~~and for $i = j$~~ hence for $u \perp v_1$, $d^2 f_x(u, v_1) = d\sigma_x(u)df_x(v_1) + d\sigma_x(u)df_x(u)$

$$\cancel{(5.2) \qquad d^2 f_{\mathbf{x}}(\mathbf{v}_i, \mathbf{v}_i) = d\sigma_{\mathbf{x}}(\mathbf{v}_i)df_{\mathbf{x}}(\mathbf{v}_i) - \sum_{k\neq i} d\sigma_{\mathbf{x}}(\mathbf{v}_k)df_{\mathbf{x}}(\mathbf{v}_k).}$$

Note that so far we have not required that σ be non-constant.

Suppose now that ~~$i \neq j$~~ $u \perp v_1$. Then, multiplying equation (5.1) by $e^{-\sigma}$, we have

$$e^{-\sigma} d^2 f_{\mathbf{x}}(u, \mathbf{v}_1) + d(e^{-\sigma})_{\mathbf{x}}(u) df_{\mathbf{x}}(\mathbf{v}_1) + d(e^{-\sigma})_{\mathbf{x}}(\mathbf{v}_1) df_{\mathbf{x}}(u) = 0.$$

Differentiating again gives

$$\begin{aligned} d(e^{-\sigma})_{\mathbf{x}}(\mathbf{v}_2) d^2 f_{\mathbf{x}}(u, \mathbf{v}_1) + e^{-\sigma} d^3 f_{\mathbf{x}}(u, \mathbf{v}_1, \mathbf{v}_2) \\ + d^2(e^{-\sigma})_{\mathbf{x}}(u, \mathbf{v}_2) df_{\mathbf{x}}(\mathbf{v}_1) + d(e^{-\sigma})_{\mathbf{x}}(u) d^2 f_{\mathbf{x}}(\mathbf{v}_1, \mathbf{v}_2) \\ + d^2(e^{-\sigma})_{\mathbf{x}}(\mathbf{v}_1, \mathbf{v}_2) df_{\mathbf{x}}(u) + d(e^{-\sigma})_{\mathbf{x}}(\mathbf{v}_1) d^2 f_{\mathbf{x}}(u, \mathbf{v}_2) = 0. \end{aligned}$$

Now the second, fourth, fifth and the sum of the first and sixth terms are symmetric in v_1 and v_2; therefore, since the right hand side (trivially) is symmetric in v_1 and v_2 the third term must also be symmetric in v_1 and v_2. Thus, for ~~distinct i, j, k,~~ fixed u, for any v_1, v_2 both $\perp u$,

$$d^2(e^{-\sigma})_{\mathbf{x}}(u, \mathbf{v}_2) df_{\mathbf{x}}(\mathbf{v}_1) = d^2(e^{-\sigma})_{\mathbf{x}}(u, \mathbf{v}_1) df_{\mathbf{x}}(\mathbf{v}_2).$$

Choosing v_1, v_2 lin. indep., ~~Since $k \neq i$~~, it follows that $df_{\mathbf{x}}(\mathbf{v}_1)$ and $df_{\mathbf{x}}(\mathbf{v}_2)$ are linearly independent, and hence

$$d^2(e^{-\sigma})_{\mathbf{x}}(\mathbf{u}, \mathbf{v}) = 0$$

for all orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$. Note that we have used $n \geq 3$ in our argument. Now, by Theorem 3.7,

$$d^2(e^{-\sigma})_{\mathbf{x}}(\mathbf{u}, \mathbf{v}) = \alpha \langle \mathbf{u}, \mathbf{v} \rangle.$$

Thus it remains only to show that α is a constant; but this is relatively easy. First, by differentiation,

$$d^3(e^{-\sigma})_{\mathbf{x}}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = d\alpha_{\mathbf{x}}(\mathbf{w}) \langle \mathbf{u}, \mathbf{v} \rangle;$$

notice that f must have derivatives of order 4. Interchanging $\mathbf{u}$ and $\mathbf{w}$, we have

$$\langle d\alpha_{\mathbf{x}}(\mathbf{w})\mathbf{u} - d\alpha_{\mathbf{x}}(\mathbf{u})\mathbf{w}, \mathbf{v} \rangle = 0$$

for any $\mathbf{v}$. Therefore $d\alpha_{\mathbf{x}}(\mathbf{w})\mathbf{u} - d\alpha_{\mathbf{x}}(\mathbf{u})\mathbf{w} = 0$, but choosing $\mathbf{u}$ and $\mathbf{w}$ independent we see that $d\alpha_{\mathbf{x}}(\mathbf{w}) = 0$ for any $\mathbf{w}$, and hence that α is a constant. □

We now state and prove our main theorem.

Theorem 5.5. *Let f be a one-to-one C^4 conformal map of an open set $U \subset \mathbb{R}^n$ onto $f(U)$, and suppose that $n \geq 3$. Then f is a composition of similarities and inversions.*