The Collatz Conjecture Proof

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Abstract

The Collatz Conjecture, first posed in 1937 by Lothar Collatz, has finally been confirmed through a series of nested proofs by fifteen-year-old Cody T. Dianopoulos.

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1 Introduction

In 1937, Lothar Collatz posed a relevant additive number theory. He speculated that given a number $n \in \mathbb{N}$ can be transformed recursively to become 1 with the following series of transformations:

$$c(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

For example, start with the number 18. 18→9→28→14→7→22→11→34→17→52→26→13→40→20→10→5→16→8→4→2→1. The conjecture, which later became known as the Collatz Conjecture, states:

> This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Previously, there are no solid proofs that his conjecture was correct, although much evidence supports it. For example, computer algothims have confirmed that if a non-reducible number existed, it would have to be larger than the 5.764×10^{18} already tested values of n. Fortunately, my proof is here to confirm that no values of n are non-reducible.

2 What Needs to Be Proven?

The first step in proving the Collatz Conjecture would be to evaluate what actually *needs* to be proven. The recursive function needs to be proven across all $n \in \mathbb{N}$. Since any even numbers will just be divided by 2 enough times to transform into an odd number, this conjecture only needs to be proven for all *odd* natural numbers. This will be proven by working backwards. This whole proof is based on the fact that

$$\frac{4^k - 1}{3} \in \mathbb{N} \forall k \in \mathbb{N}$$

3 Considerations

Consider functions $o(x) = \frac{x-1}{3}$ and $e(x) = x(2^k)$ for an arbitrary $k \in \mathbb{N}$. These are the reverse of the two transformations requested in this problem. $o(x)$ converts *even inputs only* into odd outputs. $e(x)$ converts *odd inputs only* into even outputs. This means that the only senisble compound functions would be $o(e(x))$ and $e(o(x))$. Since I'm only trying to prove this for odd numbers only,

$$\text{let } f(x) = o(e(x)) = \frac{x(2^k) - 1}{3}$$

and

$$\text{let } f^a(x) = f \text{ compounded within itself } a \text{ times}$$

Also, let $\mu =$ the set of all numbers reducible to 1 through the Collatzian algorithm $c(n)$.

4 Proof

Lemma 1. *A power of 2 plus that natural power plus 1 of 2 is divisible by 3.*

Proof.
$$2^k + 2^{k+1} = 2^k(1+2) = 2^k(3) \text{ where } k \in \mathbb{N}$$
□

Lemma 2. *A power of 2, minus 1, and divided by 3 composite if and only if it is an even power of 2.*

Proof. Consider a formula for a power of 2, minus 1, and divided by 3:

$$\frac{2^k - 1}{3} = \sum_{i=0}^{k-1} \frac{2^i}{3}$$

If this were true, then it can be rearranged to be

$$2^k = 1 + \sum_{i=0}^{k-1} 2^i$$

You can use mathematical induction to show that this formula applies for $k+1$:

$$2^{k+1} = 2 + \sum_{i=0}^{k-1} 2^{i+1} = 2 + \sum_{i=1}^{k} 2^i = 1 + \sum_{i=0}^{k} 2^i$$

Because of Lemma 1, $\frac{2^i}{3}$ must be added an even amount of times, thus k must be a multiple of 2. □

Working backwards manages to state that near the end of the series of transformations, either μ must be a power of 2 initially, or will contain a number n such that
$$n = \frac{4^k - 1}{3}$$

This series, in which $a_k = \frac{4^k-1}{3}$, constitutes the first values of μ.

Lemma 3.

$$f^a(n) = \begin{cases} \frac{n-1}{3} + n \sum_{i=0}^{k-1} 4^i & \textit{if } n \equiv 1 \pmod 3 \\ \frac{2n-1}{3} + 2n \sum_{i=0}^{k-1} 4^i & \textit{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. The values of μ can be grouped into three groups: those divisible by 3, those with a remainder of 1 when divided by 3, and those with a remainder of 2 when divided by 3. Since the proof is sticking with the odd domain, $3x$ will represent the multiple of 3, $3x+4$ will be the value with remainder 1, and $3x+2$ will be the value with remainder 2.

When performing $f(3x)$, it would not be in a range of natural numbers because $\frac{3x(2^k)-1}{3} = x(2^k) - \frac{1}{3}$. This means that if the reverse sequence reaches a multiple of 3, the sequence terminates at that path.

With $f(3x+4)$, the sequence continues for $f^{2a}(x)$. This is given by the equation:

$$\frac{n(4^k)-1}{3}$$

. This sequence can be generalized to be equal to another:

$$\frac{n(4^k)-1}{3} = \frac{n-1}{3} + n\sum_{i=0}^{k-1} 4^i$$

It can be rearranged and proven through Lemma 2:

$$4^k = 1 + 3\sum_{i=0}^{k-1} 4^i \Rightarrow \frac{2^{2n}-1}{3} = \sum_{i=0}^{k-1} 4^i$$

With $f(3x+1)$, the sequence continues for $f^{2a+1}(x)$. This is given by the equation:

$$\frac{2n(4^k)-1}{3}$$

. This sequence can be generalized to be equal to another:

$$\frac{2n(4^k)-1}{3} = \frac{2n-1}{3} + 2n\sum_{i=0}^{k-1} 4^i$$

It, too, can be rearranged and proven through Lemma 2:

$$4^k = 1 + 3\sum_{i=0}^{k-1} 4^i \Rightarrow \frac{2^{2n}-1}{3} = \sum_{i=0}^{k-1} 4^i$$

Therefore,

$$f^a(n) = \begin{cases} \frac{n-1}{3} + n\sum_{i=0}^{k-1} 4^i & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n-1}{3} + 2n\sum_{i=0}^{k-1} 4^i & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$\square$

Lemma 4. *μ contains the set of all natural numbers.*

Proof. Lemma 3 states that μ now encapsulates all

$$f^a(n) = \begin{cases} \frac{n-1}{3} + n\sum_{i=0}^{k-1} 4^i & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n-1}{3} + 2n\sum_{i=0}^{k-1} 4^i & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Since $\sum_{i=0}^{k} 4^i$ has no limit, it can be disregarded and state that μ encapsulates all

$$f^a(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n-1}{3} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

This means that all that is left to do is to state that $f^a(n)$ is valid for the set of all odd natural numbers. Since $f^a(n)$ accounts for modular values of odd numbers, and $f^a(n)$ extends and terminates at all multiples of 3, $f^a(n)$ has a value for all natural numbers. $\square$

5 References

[1] Motta, Francis C., Henrique R. De Oliveira, and Thiago A. Catalan. An Analysis of the Collatz Conjecture. Print.