Human-understandable proof of $650 \implies x \diamond y = x$

Bruno Le Floch

May 19, 2025

Theorem 1 (650 $\implies$ 4)**.** *Equation 650* $x = x \diamond (y \diamond ((z \diamond x) \diamond y))$ *implies Equation 4* $x = x \diamond y$.

Throughout the proof, free variables in the various equations are arbitrary elements of the magma, for instance $(y \diamond x) \diamond x = y \diamond x$ stands for $\forall x \, \forall y \, (y \diamond x) \diamond x = y \diamond x$, unless a particular assignment has been specified (in terms of other variables). We introduce the notation

$$f(x, y) = x \diamond (y \diamond x), \qquad g(x, y) = f(x, y \diamond x) = x \diamond ((y \diamond x) \diamond x), \tag{1}$$

$$x^2 = x \diamond x, \qquad x^3 = x^2 \diamond x, \qquad x_3 = x \diamond x^2. \tag{2}$$

The proof consists of showing many intermediate results of the form $A \diamond B = A$ where A and B are various expressions, with the aim of making them more and more decoupled from each other as the proof progresses. We begin with five easy identities involving simply f and/or g with no nesting, used throughout the proof. Then we show idempotence $f(u, v)^2 = f(u, v)$ for suitable v in (8), leading us to the result $(x \diamond y) \diamond g(x, z) = x \diamond y$ (9). We then build towards a proof of Equation 58, $x \diamond (y \diamond (z \diamond x)) = x$, through a collection of intermediate results that are then enough to prove Equation 4, $x = x \diamond y$, and finish the proof.

Lemma 1 (Easy identities on products of f and g)**.**

$$x \diamond f(y, z \diamond x) = x, \tag{3}$$

$$x \diamond (z \diamond x)_3 = x, \tag{4}$$

$$f(x, y \diamond w) \diamond f(z, w) = f(x, y \diamond w), \tag{5}$$

$$y \diamond w = y \implies f(x, y) \diamond f(z, w) = f(x, y), \tag{6}$$

$$g(x, y) \diamond f(z, x) = g(x, y). \tag{7}$$

Proof. The first identity is just Equation 650 in disguise. The second is obtained by taking $y = z \diamond x$ in the first and noting that $f(z \diamond x, z \diamond x) = (z \diamond x)_3$. The third is the main identity with $(x, y, z) \to (f(x, y \diamond w), z, w)$ and using $w \diamond f(x, y \diamond w) = w$ to simplify the last part of the right-hand side. The fourth is a rewriting of the third, which proves very useful in the following, as it tells us the product of two rather general $f(\_,\_)$. The fifth is a special case of the third with $w = x$, using that $f(x, y \diamond x) = g(x, y)$. It allows us to rewrite some products of $g(\_,\_)$ times $f(\_,\_)$. □

Lemma 2 (Projection for multiplication by g)**.**

$$f(v, w) = v \implies f(u, v)^2 = f(u, v), \tag{8}$$

$$(x \diamond y) \diamond g(x, z) = x \diamond y. \tag{9}$$

Proof. Select some arbitrary element a_0 of the magma (say, $a_0 = u$) and calculate. In the first two steps we manipulate the second operand of $f(u, v) \diamond f(u, v)$ to convert it to $f(f(u, v), \_)$, then we apply (6) to the resulting product of f. Specifically, the first application of (6) uses that $f(v, w) = v$, and the second one that $v \diamond f(a_0, w \diamond v) = v$ by the main identity (3):

$$\begin{aligned} f(u, v)^2 &\overset{(6)}{=} f(u, v) \diamond \big(f(u, v) \diamond f(a_0, w \diamond v)\big) \\ &\overset{(5)}{=} f(u, v) \diamond \Big(f(u, v) \diamond \big(f(a_0, w \diamond v) \diamond f(u, v)\big)\Big) \overset{(6)}{=} f(u, v). \end{aligned} \tag{10}$$

We now establish (9). Denote $u = f(y, x)$, $v = g(x, z)$ and $w = (z \diamond x) \diamond x$. By (7), $v = v \diamond f(\_, x)$. We use this in two ways. First, since $u = f(y, x)$ we have $v = v \diamond u$. Second, we get $v = v \diamond f(w, x) = v \diamond (w \diamond v) = f(v, w)$, which serves as a premise to (8), so that $f(u, v)^2 = f(u, v)$. Simplifying $f(u, v) = u \diamond (v \diamond u) = u \diamond v$, we have the idempotence $(u \diamond v)^2 = u \diamond v$. We insert this property into (4) to get $v = v \diamond (u \diamond v)_3 = v \diamond (u \diamond v) = f(v, u)$. It is then easy to conclude:

$$(x \diamond y) \diamond v = (x \diamond y) \diamond f(v, u) \overset{(3)}{=} x \diamond y, \tag{11}$$

where in the second step we used that $u = f(y, x) = y \diamond (x \diamond y)$. □

Lemma 3 (Towards Equation 58)**.**

$$x \diamond (g(y, z) \diamond (y \diamond x)) = x, \qquad \textit{(restricted version of equation 58)}, \tag{12}$$

$$x \diamond \big((y \diamond (z \diamond x)) \diamond (z \diamond x)\big) = x, \qquad \textit{(equation 58 with duplicated } z \diamond x\textit{)}, \tag{13}$$

$$y \diamond (z \diamond x) = y \implies x \diamond y = x, \qquad \textit{(equation 4 under a condition)}, \tag{14}$$

$$(y \diamond x) \diamond x = y \diamond x, \qquad \textit{(right-multiplication idempotent)}, \tag{15}$$

$$x^3 = x^2, \qquad \textit{(cubes are squares)}, \tag{16}$$

$$x \diamond (y \diamond (z \diamond x)) = x, \qquad \textit{(equation 58)}, \tag{17}$$

Proof. Our main aim here is to prove Equation 58, (17). We begin with a special case (12) involving g instead of a general element of the magma, then a version of the equation where $z \diamond x$ is duplicated. It is then simplified using a convenient deduplication identity (15) (Equation 378).

Let us begin. For the first identity, we use (9) with x, y swapped to transform the product of $g(y, z)$ and $y \diamond x$ into a call to f, and we then simplify using the main equation (3),

$$x \diamond \big(g(y, z) \diamond (y \diamond x)\big) \overset{(9)}{=} x \diamond \Big(g(y, z) \diamond \big((y \diamond x) \diamond g(y, z)\big)\Big) \overset{(3)}{=} x. \tag{18}$$

The second one is a more tricky calculation. The basic idea is that (9) and (12) both involve the product of some $w \diamond u$ and $g(w, y)$ in different orders, so we can get leverage from making these two factors equal: set $u = (y \diamond w) \diamond w$ so that $w \diamond u = g(w, y)$. One gets

$$u \overset{(12)}{=} u \diamond (w \diamond u)^2 \overset{(9)}{=} u \diamond (w \diamond u) = f(u, w). \tag{19}$$

Then, by taking $w = z \diamond x$ and applying the main equation, we deduce

$$x \diamond u = x \diamond f(u, z \diamond x) = x, \qquad \text{for } u = \big(y \diamond (z \diamond x)\big) \diamond (z \diamond x), \tag{20}$$

which is the desired identity (13).

The rest of the proof is very straightforward. If $y \diamond (z \diamond x) = y$ then the left-hand side of (13) simplifies all the way down to $x \diamond y$, thus (14) holds. Then, by (12) we have $\exists w, x \diamond (w \diamond (y \diamond x))$, so we can apply (14) with (x, y) replaced by $(y \diamond x, x)$, which yields exactly (15), the idempotence of right-multiplication. A particular case used later is that cubes are the same as squares, as stated in (16). Finally, idempotence of right-multiplication simplifies (13) to the desired Equation 58, (17). □

Lemma 4 (Equation 4)**.**

$$x \diamond y = x, \tag{21}$$

Proof. At this point, we know enough about the operation to conclude. From (9) with $z = x$ we have $(x \diamond y) \diamond g(x, x) = x \diamond y$, which takes the form of the premise of (14) applied to $(x^2, x \diamond y, x)$ since $g(x, x) = x \diamond x^3 = x \diamond x^2$ by (16). We learn that $x^2 \diamond (x \diamond y) = x^2$. We then calculate as follows,

$$x \diamond y \overset{(17)}{=} x \diamond (y \diamond (x^2 \diamond (x \diamond y))) = x \diamond (y \diamond x^2) \overset{(17)}{=} x. \tag{22}$$

□

Once we have the left-projection law (21), Equation 4, all the consequences are immediately found by direct computation, for instance Equation 448 holds, $x = x \diamond (y \diamond (z \diamond (x \diamond z)))$.