1. Equation 1526

I work with the free multiplicative group G on countably many generators as usual. The functional equation is

$$f(f(f(h)h^{-1})f(1)^{-1}) = h^{-1}f(1)^{-1}.$$

Think of this as saying that if $(1, x_1), (a, b), (ba^{-1}, c) \in f$, then we must have $(cx_1^{-1}, a^{-1}x_1^{-1}) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

1. E is finite.
2. E is a function.
3. $(1, x_1), (x_1, x_2), (x_2x_1^{-1}, x_1^{-1}), (x_1^{-2}, x_1^{-2}) \in E$ (where x_1, x_2 are distinct generator).
4. If $(a, b), (ba^{-1}, c) \in E$, then $(cx_1^{-1}, a^{-1}x_1^{-1}) \in E$.
5. If $(a, b), (c, d) \in E$ and $ab^{-1} = cd^{-1}$, then $(a, b) = (c, d)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a, b) \in E$ for some $b \in G$.

If $ba^{-1} \in \mathrm{dom}(E)$, then by condition (4) we are already done. So reduce to the case when $ba^{-1} \notin \mathrm{dom}(E)$. Let c be any generator not appearing anywhere in E, and fix

$$E' := E \cup \{(ba^{-1}, c), (cx_1^{-1}, a^{-1})x_1^{-1}\}.$$

Conditions (1), (2), and (3) on E' are clear, as is condition (5) from the newness of c.

For condition (4), we need to note that $a \neq 1$ since $ba^{-1} \notin \mathrm{dom}(E)$. Thus, from (5) for E, we have $ba^{-1} \neq x_1$. A case analysis now suffices (using (5) to show that we don't cause any problems with the old pairs in E).

Case 2: Assume $a \notin \mathrm{dom}(E)$. If there is some $(x, y) \in E$ with $a = yx^{-1}$, then use Case 1 to make the functional equation hold for x, and then a will belong to the domain of that extension, and we reduce to Case 1 again.

So, we may assume there is not such pair (x, y). Taking b to be any generator of G not appearing in E or in a, then $E \cup \{(a, b)\} \in \mathscr{E}$ and we again revert to Case 1. $\square$

With the choice that $f(1) = x_1$, the functional equation for 1223 is

$$f(f(x_1^{-1}f(x_1^{-1})^{-1})f(x_1^{-1})x_1) = 1.$$

The initial seed

$$\{(1, x_1), (x_1, x_2), (x_2x_1^{-1}, x_1^{-1}), (x_1^{-2}, x_1^{-2}), (x_1^{-1}, x_3), (x_1^{-1}x_3^{-1}, x_4), (x_4x_3x_1, x_5), (x_5x_1^{-1}, x_3)\}$$

works to contradict this equation.

The functional equation for 2744 is

$$f(x_1^{-1}f(f(h)x_1^{-1})^{-1}) = hx_1^{-1}f(f(h)x_1^{-1})^{-1}.$$

The seed

$$\{(1, x_1), (x_1, x_2), (x_2x_1^{-1}, x_1^{-1}), (x_1^{-2}, x_1^{-2}), (x_3, x_4), (x_4x_1^{-1}, x_5), (x_1^{-1}x_5^{-1}, x_6)\}$$

works.