EQUIVALENT NOTIONS OF PRINCIPAL ELEMENT IN A QUANTALE

E. DEAN YOUNG AND HARM DERKSEN

Abstract. A quantale is an abstraction of the notion of an ideal in a ring. After establishing definitions intrinsic to quantales and their modules, the authors consider three definitions of principal element in a quantale, each a generalization of principal elements in a ring. Two of these definitions were considered in papers by Ward and Dilworth. The main result is a proof that these definitions are equivalent in a modular quantale. Proofs of Nakayama's Lemma and the Chinese Remainder Theorem, generalized to the setting of quantales, are included at the end.

1. Introduction

We begin by giving an exposition of the definition and basic properties of quantales in Section 2. Fundamental properties of quantales are tabulated there, along with the example of the quantale $\mathrm{End}(M)$ for a complete lattice M. Section 3 is an analogous consideration for quantale modules, which are to modules as quantales are to rings. Section 4 establishes the notion of saturated map in the category of quantale modules over a given quantale. Section 5 uses the saturated map concept to give a new notion of principal element in a quantale, which is shown to be equivalent to the notion of principal element as defined first by R. P. Dilworth in [1] in any modular quantale. Before this, Dilworth and Ward had defined a notion of principal in [2]. We show here that these definitions are in fact equivalent in a modular quantale (any quantale arising from a ring is modular). In Section 6, we give a proof of Nakayama's lemma, and Section 7 contains a proof of the Chinese Remainder Theorem, both generalized to the setting of quantales.

2. Quantales

A morphism of partially ordered sets (posets) A and B is a function $\mathfrak{f} : A \to B$ such that $\mathfrak{a} \leq \mathfrak{b}$ implies $\mathfrak{f}(\mathfrak{a}) \leq \mathfrak{f}(\mathfrak{b})$. If a poset has binary meets and joins we say it is a lattice. A lattice with arbitrary meets and joins is called complete. For a complete lattice A, we write $\sum_{i\in I} \mathfrak{a}_i$

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for the join of elements $\{\mathfrak{a}_i\}_{i\in I}$ in A and $\mathfrak{a}+\mathfrak{b}$ for the join of elements $\mathfrak{a}$ and $\mathfrak{b}$ in A. Similarly, we write $\bigcap_{i\in I}\mathfrak{a}_i$ for the meet of elements $\{\mathfrak{a}_i\}_{i\in I}$ in X and $\mathfrak{a}\cap\mathfrak{b}$ for the meet of elements $\mathfrak{a}$ and $\mathfrak{b}$ in A. We denote the unique minimal (resp. maximal) element in complete lattice A by 0 (resp. 1).

We will use the concept of adjoint poset morphisms here.

Definition 1. For lattices A and B, a morphism $\mathfrak{f}: A\to B$ is left adjoint to a morphism $\mathfrak{g}: B\to A$, when $\mathfrak{f}(\mathfrak{a})\leq\mathfrak{b} \iff \mathfrak{a}\leq\mathfrak{g}(\mathfrak{b})$ for each $\mathfrak{a}\in A, \mathfrak{b}\in B$.

The adjoint functor theorem for lattices says that a poset morphism which preserves arbitrary joins is a left adjoint, and a poset morphism which preserves arbitrary meets is a right adjoint:

Theorem 2. *Let $\mathfrak{f}: A\to B$ be a morphism of complete lattices such that*

$$\mathfrak{f}\left(\sum_{i\in I}\mathfrak{a}_i\right)=\sum_{i\in I}\mathfrak{f}(\mathfrak{a}_i)$$

for any (possibly empty) indexing set I, and $\mathfrak{a}_i\in A$. Then $\mathfrak{f}$ has a left adjoint $\mathfrak{g}: B\to A$ given by

$$\mathfrak{g}(\mathfrak{b})=\sum_{\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}}\mathfrak{c}.$$

Proof. Take $\mathfrak{a},\mathfrak{b}\in A$. Then we have

$$\mathfrak{a}\leq\mathfrak{g}(\mathfrak{b})=\sum_{\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}}\mathfrak{c}\Rightarrow\mathfrak{f}(\mathfrak{a})\leq\mathfrak{f}\left(\sum_{\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}}\mathfrak{c}\right)=\sum_{\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}}\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}$$

and

$$\mathfrak{f}(\mathfrak{a})\leq\mathfrak{b}\Rightarrow\mathfrak{a}\leq\sum_{\mathfrak{f}(\mathfrak{c})\leq\mathfrak{b}}\mathfrak{c}=\mathfrak{g}(\mathfrak{b}).$$

□

Similarly, a morphism of complete lattices $\mathfrak{g}: B\to A$ which preserves (set indexed) meets has a left adjoint $\mathfrak{f}: A\to B$ defined where $\mathfrak{f}(\mathfrak{a})=\bigcap_{\mathfrak{g}(\mathfrak{b})\geq\mathfrak{a}}\mathfrak{b}$. The proof is entirely analogous.

Definition 3. A quantale is a triple $(E,\leq,*)$ where $(E,\leq)$ is a complete lattice, $(E,*)$ is a monoid (we will write $\mathfrak{a}\mathfrak{b}$ for $\mathfrak{a}*\mathfrak{b}$), and where left and right multiplication distributes over arbitrary joins (including the empty join): $\mathfrak{a}\sum_{i\in I}\mathfrak{a}_i=\sum_{i\in I}\mathfrak{a}\mathfrak{a}_i$ and $\left(\sum_{i\in I}\mathfrak{a}_i\right)\mathfrak{a}=\sum_{i\in I}\mathfrak{a}_i\mathfrak{a}$, for $\mathfrak{a},\{\mathfrak{a}_i\}_{i\in I}$ in E.

We denote the product of $\mathfrak{a}, \mathfrak{b} \in E$ as $\mathfrak{a}\mathfrak{b}$ and the product of elements $\{\mathfrak{a}_i\}_{i=1}^n$ of elements of E as $\Pi_{i=1}^n \mathfrak{a}_i$. We write 1 for the greatest element of E and 0 for the least element.

Definition 4. A morphism $\mathfrak{f} : E \to F$ of quantales $(E, \leq, *)$ and $(F, \leq, *)$ is a pair $(\mathfrak{f}_*, \mathfrak{f}^*)$, where $\mathfrak{f}_* : E \to F$ is a left adjoint poset morphism, with $\mathfrak{f}^* : F \to E$ its right adjoint, such that $\mathfrak{f}_* : (E, *) \to (F, *)$ is a morphism of monoids. We may define $\mathfrak{f}$ by defining its left adjoint $\mathfrak{f}_*$ alone.

The category of sup-lattices is the category whose objects are complete lattices, and whose morphisms are join preserving functions $\mathfrak{f} : A \to B$ such that $\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{f}(\mathfrak{a}) \leq \mathfrak{f}(\mathfrak{b})$. Quantales can equivalently be defined as monoid objects in the category of sup-lattices. There is a direct correspondence between join preserving complete lattice maps and meet preserving complete lattice maps, and indeed the category of sup-lattices is isomorphic to the opposite category of complete lattices with meet preserving poset maps. For a left adjoint morphism $\mathfrak{f}_* : A \to B$ of complete lattices, we write $\mathfrak{f}^* : B \to A$ for its unique right adjoint. For a morphism $\mathfrak{f} : E \to F$ of quantales, we can write $\mathfrak{f}_* : E \to F$ for $\mathfrak{f}$ and $\mathfrak{f}^* : F \to E$ for its corresponding right adjoint lattice morphism.

It is readily seen that there is a left adjoint embedding functor from monoids to quantales, which sends a monoid M to the triple $(\mathcal{P}(M), \leq, *)$, where $\mathcal{P}(M)$ is the power set of M, $\leq$ is inclusion of subsets, and $X * Y = \{xy : x \in X, y \in Y\}$. However the functor that relates to our own interests is from rings to quantales; every (commutative) ring A induces a quantale A^{quant} as the triple $(E, \subseteq, *)$, where E is the set of ideals of A, $\subseteq$ is inclusion of ideals, and $*$ is product of ideals. Join becomes sum of ideals, meet becomes intersection of ideals, and the monoidal operation becomes ideal product. Every morphism of rings $f : A \to B$ induces a morphism of quantales $f^{\text{quant}} : A^{\text{quant}} \to B^{\text{quant}}$, which is a left adjoint morphism sending an ideal $\mathfrak{a} \in A$ to $\mathfrak{a}B$ in B, whose right adjoint $(f^{\text{quant}})^* : B^{\text{quant}} \to A^{\text{quant}}$ sends an ideal $\mathfrak{b}$ in B to $\mathfrak{f}^*(\mathfrak{b})$.

For a quantale E with element $\mathfrak{a} \in E$, we write $\lfloor \mathfrak{a} \rfloor$ for the set $\{\mathfrak{b} \in E : \mathfrak{b} \geq \mathfrak{a}\}$. We can define a multiplication operation on $\lfloor \mathfrak{a} \rfloor$ making it into a quantale in its own right, where the complete lattice structure is inherited from E, and where we take $\mathfrak{b} * \mathfrak{c} = \mathfrak{b}\mathfrak{c} + \mathfrak{a}$. Similarly, we write $\lceil \mathfrak{a} \rceil$ for the complete lattice $\{\mathfrak{b} \in E : \mathfrak{b} \leq \mathfrak{a}\}$ with partial order inherited from E. For a morphism $\mathfrak{f} : E \to F$ of quantales E and

F, we define its kernel $\ker(\mathfrak{f})$ as $\lceil \mathfrak{f}^*(0) \rceil$, its cokernel $\mathrm{coker}(\mathfrak{f})$ as $\lfloor \mathfrak{f}_*(1) \rfloor$, its image $\mathrm{Im}(\mathfrak{f})$ as $\lceil \mathfrak{f}_*(1) \rceil$, and its coimage $\mathrm{Coim}(\mathfrak{f})$ as $\lfloor \mathfrak{f}^*(0) \rfloor$.

Example 5. Let E be a complete lattice. The set $\mathrm{End}(E)$ of left adjoint morphisms of complete lattices $E \to E$ forms a monoid under composition. $\mathrm{End}(E)$ also has a partial order where, for $\mathfrak{f}, \mathfrak{g} \in \mathrm{End}(E)$, $\mathfrak{f} \leq \mathfrak{g}$ when $\mathfrak{f}_*(\mathfrak{a}) \leq \mathfrak{g}_*(\mathfrak{a})$ and $\mathfrak{f}^*(\mathfrak{a}) \geq \mathfrak{g}^*(\mathfrak{a})$ for all $\mathfrak{a} \in E$. $\mathrm{End}(E)$ is complete under this partial order. To show this it suffices to show that $\mathrm{End}(E)$ has arbitrary joins, since any poset with arbitrary joins has arbitrary meets.

Let $\{\mathfrak{f}_i\}_{i \in I}$ be elements of $\mathrm{End}(E)$. Define $\mathfrak{f}_*(\mathfrak{a}) = \sum_{i \in I} (\mathfrak{f}_i)_*(\mathfrak{a})$ and $\mathfrak{f}^*(\mathfrak{a}) = \bigcap_{i \in I} (\mathfrak{f}_i)^*(x)$. Observe that

$$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \sum_{i \in I} (\mathfrak{f}_i)_*(\mathfrak{a}) \leq \sum_{i \in I} (\mathfrak{f}_i)_*(\mathfrak{b})$$

and

$$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \bigcap_{i \in I} (\mathfrak{f}_i)^*(\mathfrak{a}) \leq \bigcap_{i \in I} (\mathfrak{f}_i)^*(\mathfrak{b}).$$

We show that $\mathfrak{f}_*$ is left adjoint to $\mathfrak{f}^*$. It suffices to show $\mathfrak{f}_*(\mathfrak{a}) \leq \mathfrak{b} \Leftrightarrow \mathfrak{a} \leq \mathfrak{f}^*(\mathfrak{b})$. We have

$$\begin{aligned} & \sum_{i \in I} (\mathfrak{f}_i)_*(\mathfrak{a}) \leq \mathfrak{b} \\ \Leftrightarrow \;\; & (\mathfrak{f}_i)_*(\mathfrak{a}) \leq \mathfrak{b} \text{ for all } i \in I \\ \Leftrightarrow \;\; & \mathfrak{a} \leq (\mathfrak{f}_i)^*(\mathfrak{b}) \text{ for all } i \in I \\ \Leftrightarrow \;\; & \mathfrak{a} \leq \bigcap_{i \in I} (\mathfrak{f}_i)^*(\mathfrak{b}). \end{aligned}$$

Thus $\mathfrak{f}_*$ is left adjoint to $\mathfrak{f}^*$, proving that $\mathrm{End}(E)$ has arbitrary joins. Clearly E satisfies the distributive law. It is thus the case that $\mathrm{End}(E)$ forms a quantale, which we also denote by $\mathrm{End}(E)$. Note that the 0 element for $\mathrm{End}(E)$ is the map sending x to 0 for each $x \in E$. The greatest element in $\mathrm{End}(E)$ is Id_E.

Remark. For a quantale E with element $\mathfrak{a} \in E$, we define the map $\mu(\mathfrak{a})_* : E \to E$, where $\mathfrak{b} \mapsto \mathfrak{a}\mathfrak{b}$. $\mu(\mathfrak{a})_*$ is left adjoint for each element $\mathfrak{a}$ since it preserves joins. Its right adjoint $\mu(\mathfrak{a})^*$ where $\mu(\mathfrak{a})^*(\mathfrak{b}) = \sum_{\mathfrak{a}\mathfrak{c} \leq \mathfrak{b}} \mathfrak{c}$ corresponds to the familiar ideal quotient of rings. We write $(\mathfrak{b} : \mathfrak{a})$ for $\mu(\mathfrak{a})^*(\mathfrak{b})$. It is readily verified that there is a natural quantale map $\mu : E \to \mathrm{End}(E)$ given where $\mathfrak{a} \mapsto \mu(\mathfrak{a})_*$, which is in particular a left adjoint map of complete lattices.

We tabulate some essential properties of quantales below, all of which follow easily from the axioms alone. Take $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \{\mathfrak{a}_i\}_{i\in I}$ in a quantale E.

Table 1: Fundamental Properties of Quantales

Property	**Dual Property**
$1\mathfrak{a} = \mathfrak{a}$	$(\mathfrak{a} : 1) = \mathfrak{a}$
$(\mathfrak{a}\mathfrak{b})\mathfrak{c} = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$	$((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{c}\mathfrak{b})$
$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}\mathfrak{c} \leq \mathfrak{b}\mathfrak{c}$	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow (\mathfrak{c} : \mathfrak{a}) \geq (\mathfrak{c} : \mathfrak{b})$
$\mathfrak{b} \leq \mathfrak{c} \Rightarrow \mathfrak{a}\mathfrak{b} \leq \mathfrak{a}\mathfrak{c}$	$\mathfrak{b} \leq \mathfrak{c} \Rightarrow (\mathfrak{b} : \mathfrak{a}) \leq (\mathfrak{c} : \mathfrak{a})$
$\mathfrak{a}\mathfrak{b} \leq \mathfrak{a}$	$\mathfrak{a} \leq (\mathfrak{a} : \mathfrak{b})$
$(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \leq \mathfrak{a}$	$\mathfrak{a} \leq (\mathfrak{a}\mathfrak{b} : \mathfrak{b})$
$\sum_{i\in I}(\mathfrak{a}\mathfrak{a}_i) = \mathfrak{a}\left(\sum_{i\in I}\mathfrak{a}_i\right)$	$\bigcap_{i\in I}(\mathfrak{a}_i : \mathfrak{a}) = \left(\bigcap_{i\in I}\mathfrak{a}_i : \mathfrak{a}\right)$
$\mathfrak{a}0 = 0$	$(\mathfrak{a} : 1) = \mathfrak{a}$
$\mathfrak{a}\left(\bigcap_{i\in I}\mathfrak{a}_i\right) \leq \bigcap_{i\in I}\mathfrak{a}\mathfrak{a}_i$	$\bigcap_{i\in I}(\mathfrak{a} : \mathfrak{a}_i) \leq \left(\mathfrak{a} : \sum_{i\in I}\mathfrak{a}_i\right)$
$\left(\sum_{i\in I}\mathfrak{a}_i\right)\mathfrak{a} = \sum_{i\in I}(\mathfrak{a}_i\mathfrak{a})$	$\left(\mathfrak{a} : \sum_{i\in I}\mathfrak{a}_i\right) = \bigcap_{i\in I}(\mathfrak{a} : \mathfrak{a}_i)$
$0\mathfrak{a} = 0$	$(\mathfrak{a} : 0) = 1$

A quantale E is called commutative if $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ for all $\mathfrak{a}, \mathfrak{b} \in E$. For the purpose of this paper, all quantales are assumed to be commutative. For more information on basic quantale theory, the reader is referred to [4].

3. Quantale Modules

Let E be a quantale.

Definition 6. A quantale module over E, or E-quantale module, is a triple $M = (X, \leq, \mu)$, where $(X, \leq)$ is a complete poset, and $\mu : E \to \mathrm{End}(X)$ is a morphism of quantales, called the structure map, with $\mathrm{End}(X)$ as in Example 5.

We write 1_M for the greatest element in M, and 0 for the least element of M. We write $\mu(\mathfrak{a})_*$ for $\mu(\mathfrak{a})$, and $\mu(\mathfrak{a})^*$ for the right adjoint corresponding to $\mu(\mathfrak{a})$. We write $\mathfrak{a}x$ for $\mu(\mathfrak{a})_*(x)$ and $(x : \mathfrak{a})$ for $\mu(\mathfrak{a})^*(x)$. By the adjoint functor theorem, $(x : \mathfrak{a}) = \sum_{\mathfrak{a}y\leq x} y$ and $\mathfrak{a}x = \bigcap_{(y:\mathfrak{a})\geq x} y$. Note that each quantale E can be viewed as a quantale module over itself with structure map $\mu : E \to \mathrm{End}(E)$ where $\mu(\mathfrak{a})_*(\mathfrak{b}) = \mathfrak{a}\mathfrak{b}$.

For an E-quantale module M with $x \in M$, we write $\lfloor x \rfloor$ for the set $\{y \in M : y \geq x\}$. We can make $\lfloor x \rfloor$ into an E-quantale module in its own right, where the complete lattice structure is inherited from M, and where take $\mathfrak{a} * y = \mathfrak{a}y + x$. Similarly, we write $\lceil x \rceil$ for the complete lattice $\{y \in M : y \leq x\}$ with partial order inherited from M. We may make this lattice into a quantale module where $\mathfrak{a} * y = \mathfrak{a}y$ for $y \leq x$.

Definition 7. A morphism $\mathfrak{f} : M \to N$ of E-quantale modules M and N is a join preserving morphism of complete lattices $\mathfrak{f}_* : M \to N$ such that $\mathfrak{f}_*(\mathfrak{a}x) = \mathfrak{a}\mathfrak{f}_*(x)$ and $\mathfrak{f}^*((x : \mathfrak{a})) = (\mathfrak{f}^*(x) : \mathfrak{a})$ for each $\mathfrak{a} \in E$, $x \in M$, where $\mathfrak{f}^* : N \to M$ is the right adjoint lattice map corresponding to $\mathfrak{f}_*$.

For a morphism $\mathfrak{f} : M \to N$ of E-quantale modules, we write $\mathfrak{f}_*$ to specify the left adjoint map $\mathfrak{f}_* : E \to F$ and $\mathfrak{f}^* : F \to E$ to specify its corresponding right adjoint lattice map. We define its kernel $\ker(\mathfrak{f})$ as $\lceil \mathfrak{f}^*(0) \rceil$, its cokernel $\mathrm{coker}(\mathfrak{f})$ as $\lfloor \mathfrak{f}_*(1) \rfloor$, its image $\mathrm{Im}(\mathfrak{f})$ as $\lceil \mathfrak{f}_*(1) \rceil$, and its coimage $\mathrm{Coim}(\mathfrak{f})$ as $\lfloor \mathfrak{f}^*(0) \rfloor$.

Let M be an E quantale module for a quantale E. Take elements $\mathfrak{a}, \mathfrak{b}, \{\mathfrak{a}_i\}_{i \in I}$ in E, $x, y, \{x_i\}_{i \in I}$ in M. The following properties follow readily from the axioms:

Table 2: Fundamental Properties of Quantale Modules

Property	**Dual Property**
$\mathfrak{a} \leq \mathfrak{b} \Rightarrow \mathfrak{a}x \leq \mathfrak{b}x$	$\mathfrak{a} \leq \mathfrak{b} \Rightarrow (x : \mathfrak{b}) \leq (x : \mathfrak{a})$
$x \leq y \Rightarrow \mathfrak{a}x \leq \mathfrak{a}y$	$x \leq y \Rightarrow (x : \mathfrak{a}) \leq (y : \mathfrak{a})$.
$1x = x$	$(x : 1) = x$
$0x = 0$	$(x : 0) = 1$
$\sum_{i \in I} (\mathfrak{a}_i x) = \left(\sum_{i \in I} \mathfrak{a}_i\right) x$	$\bigcap_{i \in I} (x : \mathfrak{a}_i) = \left(x : \sum_{i \in I} \mathfrak{a}_i\right)$
$\mathfrak{a} \sum_{i \in I} x_i = \sum_{i \in I} \mathfrak{a}x_i$	$\left(\bigcap_{i \in I} x_i : \mathfrak{a}\right) = \bigcap_{i \in I} (x_i : \mathfrak{a})$
$\mathfrak{a} \bigcap_{i \in I} x_i \leq \bigcap_{i \in I} \mathfrak{a}x_i$	$\sum_{i \in I} (x_i : \mathfrak{a}) \leq \left(\sum_{i \in I} x_i : \mathfrak{a}\right)$
$x \leq (\mathfrak{a}x : \mathfrak{a})$	$\mathfrak{a}(x : \mathfrak{a}) \leq x$.
$\mathfrak{a}(\mathfrak{b}x) = (\mathfrak{a}\mathfrak{b})x$	$((x : \mathfrak{a}) : \mathfrak{b}) = (x : \mathfrak{a}\mathfrak{b})$
$\mathfrak{a}x \leq x$	$(x : \mathfrak{a}) \geq x$.
$\mathfrak{a}0 = 0$	$(1 : \mathfrak{a}) = 1$

An A-module M induces an A^{quant}-quantale module, called M^{quant}. More precisely, define $\Phi : A^{\mathrm{quant}} \to \mathrm{End}(M^{\mathrm{quant}})$, with $\mathrm{End}(M^{\mathrm{quant}})$ as in Example 5, by taking

$$\Phi(\mathfrak{a})_*(x) = \mathfrak{a}x = \left\{ \sum_{i=1}^{n} a_i y_i : a_i \in \mathfrak{a}, y_i \in x \right\}$$

and

$$\Phi(\mathfrak{a})^*(x) = (x : \mathfrak{a}) = \{y \in M : y\mathfrak{a} \leq x\}.$$

$\Phi(\mathfrak{a})_*$ and $\Phi(\mathfrak{a})^*$ are adjoint for each $\mathfrak{a} \in A^{\text{quant}}$. Indeed, $\mathfrak{a}x \leq y \Leftrightarrow x \leq (y : \mathfrak{a})$. Φ distributes over colimits, so that it is left adjoint by the adjoint functor theorem. Moreover, $\Phi(\mathfrak{a}\mathfrak{b})_*(x) = \mathfrak{a}\mathfrak{b}x = \Phi(\mathfrak{a})_*(\Phi(\mathfrak{b})_*(x))$ and

$$\Phi(\mathfrak{a}\mathfrak{b})^*(x) = (x : \mathfrak{a}\mathfrak{b}) = ((x : \mathfrak{a}) : \mathfrak{b}) = \Phi(\mathfrak{a})_*(\Phi(\mathfrak{b})_*(x)).$$

Every A-module morphism $f : M \to N$ induces an A^{quant}-quantale module morphism $f^{\text{quant}} : M^{\text{quant}} \to N^{\text{quant}}$, the familiar extension and contraction of submodules. Let M and N be modules over a ring A and $f : M \to N$ be an A-module morphism. Take $x \in M^{\text{quant}}$, $y \in N^{\text{quant}}$. $f(x) \subseteq y$ if and only if $x \subseteq f^{-1}(y)$, so extension and contraction of submodules is indeed an adjoint relationship. It should be clear that extension and contraction commute with multiplication by an element. More information on the basic theory of quantale modules can be found at [4].

4. Saturated Maps

Let $\mathfrak{f} : M \to N$ be a morphism of quantale modules over a quantale E. $\mathfrak{f}_*$ induces a map of quantale modules $\mathfrak{g}_* : \mathrm{Coim}(\mathfrak{f}) \to \mathrm{Im}(\mathfrak{f})$ which sends x to $\mathfrak{f}_*(x)$. We can consider whether $\mathfrak{g}_*$ is an isomorphism, which is analogous to the first isomorphism theorem for modules. This is in general false; when it is true, we call $\mathfrak{f}$ saturated. Similarly, we say a morphism $\mathfrak{f} : M \to N$ of E-quantale modules is saturated if the map $\mathfrak{g}_* : \mathrm{Coim}(\mathfrak{f}) \to \mathrm{Im}(\mathfrak{f})$ where $x \mapsto \mathfrak{f}_*(x)$ is an isomorphism.

Not all morphisms are saturated: for example, take the quantale $I = k^{\text{quant}}$ for a field k. Consider the I-quantale modules I and $M = \{0, x, 1\}$, where $0 < x < 1$. The map $\mathfrak{f}_* : I \to M$ where $1 \mapsto 1$ and $0 \mapsto 0$ violates the saturation property.

Definition 8. Let $\mathfrak{f} : M \to N$ be a morphism of quantale modules over a quantale E. $\mathfrak{f}$ is saturated if any of the equivalent statements are true:

(1) The induced morphism $\mathfrak{g} : \mathrm{Coim}(\mathfrak{f}) \to \mathrm{Im}(\mathfrak{f})$, where $\mathfrak{g}_*$ sends $x \in \mathrm{Coim}(\mathfrak{f})$ to $\mathfrak{f}_*(x)$ and $\mathfrak{g}^*$ sends $y \in \mathrm{Im}(\mathfrak{f})$ to $\mathfrak{f}^*(y)$ is an isomorphism of E-quantale modules.
(2) $\mathfrak{f}_*$ is injective on $\mathrm{Coim}(\mathfrak{f})$ and $\mathfrak{f}^*$ is injective on $\mathrm{Im}(\mathfrak{f})$.
(3) $\mathfrak{f}_*$ is surjective onto $\mathrm{Im}(\mathfrak{f})$ and $\mathfrak{f}^*$ is surjective onto $\mathrm{Coim}(\mathfrak{f})$.
(4) $(\mathfrak{f}_* \circ \mathfrak{f}^*)|_{\mathrm{Im}(\mathfrak{f})} = \mathrm{Id}\,|_{\mathrm{Im}(\mathfrak{f})}$ and $(\mathfrak{f}^* \circ \mathfrak{f}_*)|_{\mathrm{Coim}(\mathfrak{f})} = \mathrm{Id}\,|_{\mathrm{Coim}(\mathfrak{f})}$

Proof. Clearly (1) $\Rightarrow$ (2), and (4) $\Rightarrow$ (1). We show (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

To show (2) $\Rightarrow$ (3), take $x \in \mathrm{Coim}(\mathfrak{f})$. $\mathfrak{f}_*(x) = \mathfrak{f}_*(\mathfrak{f}^*(\mathfrak{f}_*(x)))$ since $\mathfrak{f}_*$ is left adjoint to $\mathfrak{f}^*$, so $x = \mathfrak{f}^*(\mathfrak{f}_*(x))$. Thus $\mathfrak{f}^*$ is surjective onto $\mathrm{Coim}(\mathfrak{f})$. Conversely, take $x \in \mathrm{Im}(\mathfrak{f})$. $\mathfrak{f}^*(x) = \mathfrak{f}^*(\mathfrak{f}_*(\mathfrak{f}^*(x)))$ since $\mathfrak{f}_*$ is left adjoint to $\mathfrak{f}^*$, so $x = \mathfrak{f}_*(\mathfrak{f}^*(x))$. Thus $\mathfrak{f}_*$ is surjective onto $\mathrm{Im}(\mathfrak{f})$.

To show (3) $\Rightarrow$ (4), take $x \in \mathrm{Im}(\mathfrak{f})$. Then we get $x = \mathfrak{f}_*(y)$ for $y \in N$, so

$$\mathfrak{f}_* \circ \mathfrak{f}^*(x) = \mathfrak{f}_* \circ \mathfrak{f}^* \circ \mathfrak{f}_*(y) = \mathfrak{f}_*(y) = x.$$

Thus it follows that $(\mathfrak{f}_* \circ \mathfrak{f}^*)_{\mathrm{Im}(\mathfrak{f})} = \mathrm{Id}\,|_{\mathrm{Im}(\mathfrak{f})}$. Similarly, taking $x \in \mathrm{Coim}(\mathfrak{f})$, write $x = \mathfrak{f}^*(y)$. Then we have

$$\mathfrak{f}^* \circ \mathfrak{f}_*(x) = \mathfrak{f}^* \circ \mathfrak{f}_* \circ \mathfrak{f}^*(y) = \mathfrak{f}^*(y) = x.$$

We conclude that $(\mathfrak{f}^* \circ \mathfrak{f}_*)_{\mathrm{Coim}(\mathfrak{f})} = \mathrm{Id}\,|_{\mathrm{Coim}(\mathfrak{f})}$. ☐

Theorem 9. *Let $\mathfrak{f} : E \to F$ be a morphism of quantales. Then*

(i) There is a unique $\theta : \mathrm{Coim}(\mathfrak{f}) \to \mathrm{Im}(\mathfrak{f})$ such that $\mathfrak{f}_*$ factors as $E \overset{P_{\mathfrak{f}^*(0)}}{\to} \mathrm{Coim}(\mathfrak{f}) \overset{\theta}{\to} \mathrm{Im}(\mathfrak{f}) \overset{\iota}{\to} F$ where ι is the embedding. θ is an isomorphism if and only if $\mathfrak{f}$ is saturated.
(ii) There is a unique $\theta : \mathrm{Im}(\mathfrak{f}) \to \mathrm{Coim}(\mathfrak{f})$ such that $\mathfrak{f}^*$ factors as $F \overset{I_{\mathfrak{f}^*(1)}}{\to} \mathrm{Im}(\mathfrak{f}_*) \overset{\theta}{\to} \mathrm{Im}(\mathfrak{f}_*) \overset{\iota}{\to} E$ where ι is the embedding. θ is an isomorphism if and only if $\mathfrak{f}$ is saturated.

Proof. For (i), We must define $\theta(\mathfrak{a}) = \iota(\theta(P_{\mathfrak{f}^*(0)}(\mathfrak{a}))) = \mathfrak{f}_*(\mathfrak{a})$ for $\mathfrak{a} \in \mathrm{Coim}(\mathfrak{f})$, which determines $\mathfrak{f}_*$. Then, for $\mathfrak{a} \in E$,

$$\iota \circ \mathfrak{f}_* \circ P_{\mathfrak{f}^*(0)}(\mathfrak{a}) = \lceil \mathfrak{f}_*(\mathfrak{a} + \mathfrak{f}^*(0)) \rceil = \mathfrak{f}_*(\mathfrak{a}) + \mathfrak{f}_*(\mathfrak{f}^*(0)) = \mathfrak{f}_*(\mathfrak{a}).$$

So that indeed $\mathfrak{f}_*$ factors as $\iota \circ \theta \circ P_{\mathfrak{f}^*(0)}$. That θ is an isomorphism if and only if $\mathfrak{f}$ is saturated follows from definition.

(ii) Follows similarly. ☐

Theorem 10. *Let $\mathfrak{f}_* : E \to F$ and $\mathfrak{g}_* : F \to G$ be saturated maps. Then $\mathfrak{g}_* \circ \mathfrak{f}_* : E \to G$ is saturated.*

Proof. Let $\theta : \mathrm{Coim}(\mathfrak{f}_*) \to \mathrm{Im}(\mathfrak{f}_*)$ be the isomorphism corresponding to $\mathfrak{f}_*$ as in Theorem 9. Similarly, let $\eta : \mathrm{Coim}(\mathfrak{g}_*) \to \mathrm{Im}(\mathfrak{g}_*)$ be the isomorphism corresponding to $\mathfrak{g}_*$ as in Theorem 9. Let $\theta' : \lfloor \mathfrak{f}^*(\mathfrak{g}^*(0)) \rfloor \to \{\mathfrak{a} \in F : \mathfrak{g}^*(0) \leq \mathfrak{a} \leq \mathfrak{f}_*(1)\}$ be the bijection formed by restriction of θ, and let $\eta' : \{\mathfrak{a} \in F : \mathfrak{g}^*(0) \leq \mathfrak{a} \leq \mathfrak{f}_*(1)\} \to \lceil \mathfrak{g}_*(\mathfrak{f}_*(1)) \rceil$ be the bijection formed by restriction of η. $\mathfrak{g}_* \circ \mathfrak{f}_*(\mathfrak{a}) = \eta' \circ \theta'(\mathfrak{a})$ for each $\mathfrak{a} \geq \mathfrak{f}^*(\mathfrak{g}^*(0))$, so that $\eta' \circ \theta'$ is an isomorphism corresponding to $\mathfrak{g}_* \circ \mathfrak{f}_*$ as in Theorem 9. Therefore $\mathfrak{g} \circ \mathfrak{f}$ is saturated. ☐

5. Equivalent Notions Principal Element

A central concept in the theory of quantales is the notion of a principal element, which approximates the notion of principal ideal in a ring. Principal elements were first introduced in quantales by R. P. Dilworth in [1], replacing a weaker notion of principal defined by Ward and Dilworth in [2]. These correspond to Definition 15 and Definition 16, respectively. We show that these are in fact equivalent in a modular quantale. We also provide a third definition, which has proven to be advantageous in our own work. In honor of R. P. Dilworth, we call Dilworth's notion of principal 'Dilworth principal' in the present publication. The novel definition, Definition 11, will be called principal.

Definition 11. Let M be a quantale module over a quantale E. Note that a morphism $\mathfrak{f}_* : E \to M$ of quantale modules is determined by $\mathfrak{f}_*(1)$, since $\mathfrak{f}_*(\mathfrak{a}) = \mathfrak{a}\mathfrak{f}_*(1)$, and that every element $x \in M$ determines a map $\mathfrak{f}_* : E \to M$ where $\mathfrak{f}_*(1) = x$, which is not necessarily saturated. An element $x \in M$ is called principal if the unique map $E \to M$ sending 1 to x is saturated. In particular we can take $M = E$ as a quantale module over itself to consider elements which are principal in a quantale. Note that, for a module M over a commutative ring A, a submodule in M generated by 1 element is always Dilworth principal in M^{quant}.

Definition 12. Let E be a quantale. For any $\mathfrak{a} \in E$ let $I_{\mathfrak{a}} : E \to E$ be the lattice morphism sending $\mathfrak{b} \in E$ to $\mathfrak{b} \cap \mathfrak{a}$. For each $\mathfrak{a} \in E$, let $P_{\mathfrak{a}} : E \to E$ be the lattice morphism sending $\mathfrak{b} \in E$ to $\mathfrak{b} + \mathfrak{a}$.

Definition 13. We call a quantale E modular if, for each $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in E$ with $\mathfrak{b} \leq \mathfrak{c}$, $(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + \mathfrak{b}$.

Proposition 14. *Let A be a ring. A^{quant} is modular.*

Proof. Take ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A^{\mathrm{quant}}$ with $\mathfrak{b} \leq \mathfrak{c}$. We show that $(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + \mathfrak{b}$ by showing that $(\mathfrak{a} \cap \mathfrak{c}) + \mathfrak{b} \leq (\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c}$ and

$(\mathfrak{a}+\mathfrak{b})\cap\mathfrak{c} \leq (\mathfrak{a}\cap\mathfrak{c})+\mathfrak{b}$. Firstly,

$$(\mathfrak{a}\cap\mathfrak{c})+\mathfrak{b} \leq \mathfrak{c}$$

since $\mathfrak{b} \leq \mathfrak{c}$ and $\mathfrak{a}\cap\mathfrak{c} \leq \mathfrak{c}$, and

$$(\mathfrak{a}\cap\mathfrak{c})+\mathfrak{b} \leq \mathfrak{a}+\mathfrak{b}$$

since $\mathfrak{a}\cap\mathfrak{c} \leq \mathfrak{a}$, so

$$(\mathfrak{a}\cap\mathfrak{c})+\mathfrak{b} \leq (\mathfrak{a}+\mathfrak{b})\cap\mathfrak{c}.$$

For the other inequality, take $x \in (\mathfrak{a}+\mathfrak{b})\cap\mathfrak{c}$. We may write $x = a+b$ for $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Then $a = x-b \in \mathfrak{c}$ since $x \in \mathfrak{c}$ and $b \in \mathfrak{b} \subset \mathfrak{c}$. So $a \in \mathfrak{a}\cap\mathfrak{c}$. We have expressed $x = a+b$ for $a \in \mathfrak{a}\cap\mathfrak{c}$ and $b \in \mathfrak{b}$. So $x \in (\mathfrak{a}\cap\mathfrak{c})+\mathfrak{b}$. □

Definition 15. Let E be a quantale with structure map $\mu : E \to \mathrm{End}(E)$ as in the remark after Example 5. An element $\mathfrak{p} \in E$ is Dilworth principal if for each $\mathfrak{a} \in E$, the following two diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{\mu(\mathfrak{p})_*} & E \\ \downarrow I_{(\mathfrak{a}:\mathfrak{p})} & & \downarrow I_{\mathfrak{a}} \\ E & \xrightarrow{\mu(\mathfrak{p})_*} & E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{\mu(\mathfrak{p})^*} & E \\ \downarrow P_{\mathfrak{p}\mathfrak{a}} & & \downarrow P_{\mathfrak{a}} \\ E & \xrightarrow{\mu(\mathfrak{p})^*} & E \end{array}$$

See [1] for the original exposition of this definition.

Definition 16. Let E be a quantale and set $\mu : E \to \mathrm{End}(E)$ the quantale map sending each element to the map which multiplies by that element. We say an element $\mathfrak{p} \in E$ is second Dilworth principal if $\mu(\mathfrak{p})_* \circ \mu(\mathfrak{p})^* = I_{\mathfrak{p}}$ and $\mu(\mathfrak{p})^* \circ \mu(\mathfrak{p})_* = P_{(0:\mathfrak{p})}$. This definition was first given in [2].

Theorem 17. *Let E be a modular quantale and let $\mathfrak{p} \in E$ be an element. $\mathfrak{p}$ is principal if and only if it is Dilworth principal, if and only if it is second Dilworth principal.*

Proof. We show that principal implies Dilworth principal, that Dilworth principal implies second Dilworth principal, and that second Dilworth principal implies principal.

Suppose $\mathfrak{p} \in E$ is principal. The map $\mu(\mathfrak{p})_* : E \to E$ factors through $\mathrm{Coim}(\mu(\mathfrak{p}))$ and $\mathrm{Im}(\mu(\mathfrak{p}))$ as $E \xrightarrow{P_{(0:\mathfrak{p})}} \mathrm{Coim}(\mu(\mathfrak{p})) \xrightarrow{\theta} \mathrm{Im}(\mu(\mathfrak{p})) \xrightarrow{\iota} E$ by Theorem 9, where $\iota : \mathrm{Im}(\mu(\mathfrak{p})) \to E$ is the embedding. Since $\mathfrak{p}$ is principal, $\mu(\mathfrak{p})$ is saturated, so that θ is an isomorphism again by Theorem 9. Take $\mathfrak{a} \in E$. Since $(0:\mathfrak{p}) \leq (\mathfrak{a}:\mathfrak{p})$ and E is modular, the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "P_{(0:\mathfrak{p})}"] \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] & \mathrm{Coim}(\mu(\mathfrak{p})) \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] \\
E \arrow[r, "P_{(0:\mathfrak{p})}"] & \mathrm{Coim}(\mu(\mathfrak{p}))
\end{tikzcd}$$

The following diagram also commutes since θ is an isomorphism:

$$\begin{tikzcd}
\mathrm{Coim}(\mu(\mathfrak{p})) \arrow[r, "\theta"] \arrow[d, "I_{\theta^{-1}(\mathfrak{a})}"] & \mathrm{Im}(\mu(\mathfrak{p})) \arrow[d, "I_{\mathfrak{a}}"] \\
\mathrm{Coim}(\mu(\mathfrak{p})) \arrow[r, "\theta"] & \mathrm{Im}(\mu(\mathfrak{p}))
\end{tikzcd}$$

Since $\theta^{-1}(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{p})$, the following diagram commutes for each $\mathfrak{a} \in E$:

$$\begin{tikzcd}
E \arrow[r, "P_{(0:\mathfrak{p})}"] \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] & \lfloor (0 : \mathfrak{p}) \rfloor \arrow[r, "\theta"] \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] & \lceil \mathfrak{p} \rceil \arrow[r, "\iota"] \arrow[d, "I_{\mathfrak{a}}"] & E \arrow[d, "I_{\mathfrak{a}}"] \\
E \arrow[r, "P_{(0:\mathfrak{p})}"] & \lfloor (0 : \mathfrak{p}) \rfloor \arrow[r, "\theta"] & \lceil \mathfrak{p} \rceil \arrow[r, "\iota"] & E
\end{tikzcd}$$

Thus the following commutes for each $\mathfrak{a} \in E$:

$$\begin{tikzcd}
E \arrow[r, "\mu(\mathfrak{p})_*"] \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] & E \arrow[d, "I_{\mathfrak{a}}"] \\
E \arrow[r, "\mu(\mathfrak{p})_*"] & E
\end{tikzcd}$$

To show the commutativity of the other diagram as in Definition 15, note that the map $\mu(\mathfrak{p})^* : E \to E$ factors through $\mathrm{Im}(\mu(\mathfrak{p}))$ and $\mathrm{Coim}(\mu(\mathfrak{p}))$ as $E \xrightarrow{I_{\mathfrak{p}}} \mathrm{Im}(\mu(\mathfrak{p})) \xrightarrow{\eta} \mathrm{Coim}(\mu(\mathfrak{p})) \xrightarrow{\iota} E$ for an isomorphism η in the same manner as in Theorem 9, where $\iota : \mathrm{Coim}(\mu(\mathfrak{p})) \to E$ is the embedding. Take $\mathfrak{a} \in E$. Since $\mathfrak{p}\mathfrak{a} \leq \mathfrak{p}$ and E is modular, the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "I_{\mathfrak{p}}"] \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] & \mathrm{Im}(\mu(\mathfrak{p})) \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] \\
E \arrow[r, "I_{\mathfrak{p}}"] & \mathrm{Im}(\mu(\mathfrak{p}))
\end{tikzcd}$$

The following diagram also commutes since θ is an isomorphism:

$$\begin{tikzcd}
\mathrm{Im}(\mu(\mathfrak{p})) \arrow[r, "\eta"] \arrow[d, "P_{\eta^{-1}(\mathfrak{a})}"] & \mathrm{Coim}(\mu(\mathfrak{p})) \arrow[d, "P_{\mathfrak{a}}"] \\
\mathrm{Im}(\mu(\mathfrak{p})) \arrow[r, "\eta"] & \mathrm{Coim}(\mu(\mathfrak{p}))
\end{tikzcd}$$

Since $\theta^{-1}(\mathfrak{a}) = \mathfrak{p}\mathfrak{a}$, the following diagram commutes for each $\mathfrak{a} \in E$:

$$\begin{tikzcd} E \arrow[r, "I_{\mathfrak{p}}"] \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] & \mathrm{Im}(\mu(\mathfrak{p})) \arrow[r, "\eta"] \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] & \mathrm{Coim}(\mu(\mathfrak{p})) \arrow[r, "i"] \arrow[d, "P_{\mathfrak{a}}"] & E \arrow[d, "P_{\mathfrak{a}}"] \\ E \arrow[r, "I_{\mathfrak{p}}"] & \mathrm{Im}(\mu(\mathfrak{p})) \arrow[r, "\eta"] & \mathrm{Coim}(\mu(\mathfrak{p})) \arrow[r, "i"] & E \end{tikzcd}$$

Thus the following commutes for each $\mathfrak{a} \in E$:

$$\begin{tikzcd} E \arrow[r, "\mu(\mathfrak{p})^*"] \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] & E \arrow[d, "P_{\mathfrak{a}}"] \\ E \arrow[r, "\mu(\mathfrak{p})^*"] & E \end{tikzcd}$$

Suppose next that $\mathfrak{p}$ is Dilworth principal, and let $\mathfrak{a} \in E$ be arbitrary. By definition, the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\mu(\mathfrak{p})_*"] \arrow[d, "I_{(\mathfrak{a}:\mathfrak{p})}"] & E \arrow[d, "I_{\mathfrak{a}}"] \\ E \arrow[r, "\mu(\mathfrak{p})_*"] & E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\mu(\mathfrak{p})^*"] \arrow[d, "P_{\mathfrak{p}\mathfrak{a}}"] & E \arrow[d, "P_{\mathfrak{a}}"] \\ E \arrow[r, "\mu(\mathfrak{p})^*"] & E \end{tikzcd}$$

Evalutating at 1 in the first diagram, we see that

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[d, mapsto] & \mathfrak{p} \arrow[d, mapsto] \\ (\mathfrak{a} : \mathfrak{p}) \arrow[r, mapsto] & \mathfrak{p}(\mathfrak{a} : \mathfrak{p}) = \mathfrak{p} \cap \mathfrak{a} \end{tikzcd}$$

Evaluating at 0 in the second diagram, we see that

$$\begin{tikzcd} 0 \arrow[r, mapsto] \arrow[d, mapsto] & (0 : \mathfrak{p}) \arrow[d, mapsto] \\ \mathfrak{a}\mathfrak{p} \arrow[r, mapsto] & (\mathfrak{p}\mathfrak{a} : \mathfrak{p}) = \mathfrak{a} + (0 : \mathfrak{p}) \end{tikzcd}$$

These equalities are the definition of second Dilworth principal.

Suppose lastly that $\mathfrak{p}$ second Dilworth principal, and let $\mu : E \to \mathrm{End}(E)$ be the structure map of the quantale E. Then $\mu(\mathfrak{p})_* \circ \mu(\mathfrak{p})^*|_{\mathrm{Im}(\mu(\mathfrak{p}))} = \mathrm{Id}_{\mathrm{Im}(\mu(\mathfrak{p}))}$ and $\mu(\mathfrak{p})^* \circ \mu(\mathfrak{p})_*|_{\mathrm{Coim}(\mu(\mathfrak{p}))} = \mathrm{Id}_{\mathrm{Coim}(\mu(\mathfrak{p}))}$, which is one of the equivalent statements in Definition 8. $\square$

Definition 18. Take a quantale E with element $\mathfrak{p} \in E$. If $\mathfrak{p}$ is first, second, or second Dilworth principal, we simply say $\mathfrak{p}$ is Dilworth principal.

Note that multiplication of principal elements $\mathfrak{p}, \eta \in E$ corresponds to composition of maps $\mu(\mathfrak{p})_* \circ \mu(\eta)_*$ (as is clear from Definition 11). In particular, the product of principal elements is principal.

Suppose M and N are quantale modules over a quantale E, and that $\mathfrak{p} \in M$ is Dilworth principal. For any saturated map $\mathfrak{f} : M \to N$, $\mathfrak{f}_*(\mathfrak{p})$ is Dilworth principal. Indeed, if $\mathfrak{g}_* : E \to M$ is the saturated map sending 1 to $\mathfrak{p}$, then $\mathfrak{f}_* \circ \mathfrak{g}_*$ is saturated, so that $\mathfrak{f}_*(\mathfrak{p}) = \mathfrak{f}_*(\mathfrak{g}_*(1))$ is Dilworth principal.

6. Nakayama's Lemma

Definition 19. An element $\mathfrak{m}$ of a quantale E is called maximal if $\mathfrak{m} \neq 1$ and $\mathfrak{m} < \mathfrak{a} \Rightarrow \mathfrak{a} = 1$. $\mathfrak{m} \in E$ is maximal if and only if $\lfloor \mathfrak{m} \rfloor \cong k^{\text{quant}}$, the unique quantale with two elements 0 and 1, where $0 < 1$.

Theorem 20. *Let E be a nonzero quantale such that, if $\sum_{i \in I} \mathfrak{a}_i = 1$ for $\{\mathfrak{a}_i\}_{i \in I}$ in E, then $\sum_{i \in F} \mathfrak{a}_i = 1$ for some finite set $F \subseteq I$. If $\mathfrak{a} \neq 1_E$ in E, then E has a maximal element containing $\mathfrak{a}$.*

Proof. Let $X = \{\mathfrak{b} \in E, \mathfrak{a} \subseteq \mathfrak{b}, \mathfrak{b} \neq 1\}$. We show that X has a maximal element using Zorn's lemma. First note that X is nonempty, as it contains $\mathfrak{a}$. Suppose $\mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \cdots$ is an increasing chain of elements of X. Set $\mathfrak{b} = \sum_{i \in \mathbb{N}_{\geq 0}} \mathfrak{b}_i$. $\mathfrak{b} = 1$ implies that $\sum_{i \in F} \mathfrak{b}_i = 1$ for some finite set $F \subseteq \mathbb{N}_{\geq 0}$. Choosing $i \in F$ maximal, $\mathfrak{b} = \mathfrak{b}_i \neq 1$, a contradiction. By Zorn's lemma, we conclude that E has a maximal element containing $\mathfrak{a}$. □

Theorem 21 (Nakayama's Lemma)**.** *Let E be a commutative quantale. Suppose that, if $\sum_{i \in I} \mathfrak{a}_i = 1$ for $\{\mathfrak{a}_i\}_{i \in I}$ in E, then $\sum_{i \in F} \mathfrak{a}_i = 1$ for some finite set $F \subseteq I$. Let M be a quantale module over E in which 1 is the sum of finitely many Dilworth principal elements. Write $Jac(E) = \bigcap_{\mathfrak{m} \in E, \mathfrak{m} \text{ maximal}} \mathfrak{m}$. If $Jac(E) 1_M = 1_M$ then $M = 0$.*

Proof. Suppose for a contradiction that $M \neq 0$, while $\mathrm{Jac}(E)1_M = 1_M$. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n \in M$ be Dilworth principal elements such that $1_M = \sum_{i=1}^n \mathfrak{p}_i$, with $n \in \mathbb{N}_{\geq 1}$ minimal such that this is possible. Since $1_M = \mathrm{Jac}(E)1_M$, we have

$$\sum_{i=1}^n \mathfrak{p}_i \leq \mathrm{Jac}(E) \sum_{i=1}^n \mathfrak{p}_i \leq \mathrm{Jac}(E)\mathfrak{p}_1 + \sum_{i=2}^n \mathfrak{p}_i,$$

so $\sum_{i=1}^n \mathfrak{p}_i = \mathrm{Jac}(E)\mathfrak{p}_1 + \sum_{i=2}^n \mathfrak{p}_i$. Let $y = \sum_{i=2}^n \mathfrak{p}_i$. We pass to $\lfloor y \rfloor$, writing $\mathfrak{p}$ for $\mathfrak{p}_1 + y$ in $\lfloor y \rfloor$. $\mathfrak{p}$ is Dilworth principal in $\lfloor y \rfloor$ since the canonical map from M to $\lfloor y \rfloor$ is saturated, so that the map $\mathfrak{f} : E \to \lfloor y \rfloor$ where $\mathfrak{a}$ is sent to $\mathfrak{a}\mathfrak{p}$ is injective on $\lfloor \mathfrak{a} \rfloor$, where $\mathfrak{a} = \sum_{\mathfrak{b}\mathfrak{p} \leq y} \mathfrak{b}$. If $\mathfrak{a} \neq 1_E$, then $\mathfrak{a} \leq \mathfrak{m}$ for some maximal element of E by Theorem 20. Then $\mathfrak{f}(\mathfrak{m}) \geq \mathfrak{f}(\mathrm{Jac}(E)) \geq \mathfrak{f}(1_E)$, so that $\mathfrak{m} = 1_E$ since $\mathfrak{f}$ is injective on $\lfloor \mathfrak{a} \rfloor$. Contradiction! Therefore $\mathfrak{a} = 1_E$, so that $\mathfrak{p} = 1_E \mathfrak{p} \leq y$. Thus $\mathfrak{p}_1 \leq \sum_{i=2}^{n} \mathfrak{p}_i$, contradicting the minimality of n. Thus $M = 0$. □

Corollary 22. *Let E be a quantale as in Theorem 21 and let M be an E-quantale module in which 1 is the sum of finitely many Dilworth principal elements. Take $x \in M$. If $1_M = x + \mathrm{Jac}(E)1_M$, then $x = 1_M$.*

Proof. If $1_M = x + \mathrm{Jac}(E)1_M$, then $1_{\lfloor x \rfloor} = \mathrm{Jac}(E)1_{\lfloor x \rfloor}$. Thus $\lfloor x \rfloor = 0$. We conclude $x = 1_M$. □

Corollary 23. *Let E be a quantale as in Theorem 21 and let M be an E-quantale module in which 1 is the sum of finitely many Dilworth principal elements. Take Dilworth principal elements $\{x_i\}_{i \in I}$ in M. For each $i \in I$, let y_i be the image of x_i in $\lfloor \mathrm{Jac}(E)1_M \rfloor$. If $\sum_{i \in I} y_i = 1$ in $\lfloor \mathrm{Jac}(E)1_M \rfloor$, then $\sum_{i \in I} x_i = 1$ in M.*

Proof. Suppose $\sum_{i \in I} y_i = 1$ in $\lfloor \mathrm{Jac}(E)1_M \rfloor$. Then $(\sum_{i \in I} x_i) + \mathrm{Jac}(E)1_M = \sum_{i \in I}(x_i + \mathrm{Jac}(E)1_M) = 1_M$ so $\sum_{i \in I} x_i = 1_M$ by Corollary 23. □

7. The Chinese Remainder Theorem

We show that the pairwise Chinese Remainder Theorem holds in an arbitrary quantale. Note that for two quantales E and F, one can form the product $E \,\Pi\, F$ which consists of pairs $(\mathfrak{a}, \mathfrak{b})$, ordered where $(\mathfrak{a}, \mathfrak{b}) \leq (\mathfrak{a}', \mathfrak{b}')$ when $\mathfrak{a} \leq \mathfrak{a}'$ in E and $\mathfrak{b} \leq \mathfrak{b}'$ in F, and whose multiplication is defined as $(\mathfrak{a}, \mathfrak{b})(\mathfrak{a}', \mathfrak{b}') = (\mathfrak{a}\mathfrak{a}', \mathfrak{b}\mathfrak{b}')$.

Theorem 24 (Pairwise Chinese Remainder Theorem)**.** *Let E be a quantale with elements $\mathfrak{a}, \mathfrak{b} \in E$ such that $\mathfrak{a} + \mathfrak{b} = 1$. Then $\lfloor \mathfrak{a} \cap \mathfrak{b} \rfloor \cong \lfloor \mathfrak{a} \rfloor \,\Pi\, \lfloor \mathfrak{b} \rfloor$ as quantales.*

Proof. Passing to $\lfloor \mathfrak{a} \cap \mathfrak{b} \rfloor$, it suffices to assume that $\mathfrak{a} \cap \mathfrak{b} = 0$ and show that $E \cong \lfloor \mathfrak{a} \rfloor \,\Pi\, \lfloor \mathfrak{b} \rfloor$. We show that the canonical map $\mathfrak{f} : E \to \lfloor \mathfrak{a} \rfloor \,\Pi\, \lfloor \mathfrak{b} \rfloor$ is an isomorphism. Define a set map $\mathfrak{g} : \lfloor \mathfrak{a} \rfloor \,\Pi\, \lfloor \mathfrak{b} \rfloor \to E$ where $(x, y) \mapsto$

$y\mathfrak{a} + x\mathfrak{b}$. To show $\mathfrak{f} \circ \mathfrak{g} = 1$, observe that

$$\begin{aligned}
&\mathfrak{f}(\mathfrak{g}(x, y)) \\
&= (y\mathfrak{a} + x\mathfrak{b} + \mathfrak{a}, y\mathfrak{a} + x\mathfrak{b} + \mathfrak{b}) \\
&= (x\mathfrak{b} + \mathfrak{a}, y\mathfrak{a} + \mathfrak{b}) \\
&= (x\mathfrak{b} + x\mathfrak{a} + \mathfrak{a}, y\mathfrak{a} + y\mathfrak{b} + \mathfrak{b}) \\
&= (x(\mathfrak{a} + \mathfrak{b}) + \mathfrak{a}, y(\mathfrak{a} + \mathfrak{b}) + \mathfrak{b}) \\
&= (x + \mathfrak{a}, y + \mathfrak{b}) \\
&= (x, y).
\end{aligned}$$

And for the other direction,

$\mathfrak{g}(\mathfrak{f}(x)) = \mathfrak{g}(x+\mathfrak{a}, x+\mathfrak{b}) = (x+\mathfrak{a})\mathfrak{b}+(x+\mathfrak{b})\mathfrak{a} = x\mathfrak{b}+x\mathfrak{a}+\mathfrak{a}\mathfrak{b} = x\mathfrak{b}+x\mathfrak{a} = x$ since $\mathfrak{a}\mathfrak{b} \leq \mathfrak{a} \cap \mathfrak{b} = 0$. □

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