1. Equation 3308

Equation 3308 is $x \diamond y = x \diamond (x \diamond (y \diamond x))$. Setting $y = hx$ and $a \diamond b = f(ba^{-1})a$ for some map $f\colon G \to G$, we get the functional equation

$$f^2(f(h^{-1})h) = f(h).$$

Think of this as saying that once we know $(a,b), (a^{-1}, c) \in f$, then there is some $d \in G$ such that $(ca, d), (d, b) \in f$. Of course, by symmetry in the inverses, there is also some $e \in G$ such that $(ba^{-1}, e), (e, c) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

1. E is finite.
2. E is a function with $1 \in \mathrm{dom}(E)$.
3. If $(a,b), (a^{-1}, c) \in E$, then $(ca, d), (d, b), (ba^{-1}, e), (e, c) \in E$ for some $d, e \in G$.
4. If $(a,b) \in E$ and $a^{-1} \notin \mathrm{dom}(E)$, then $ba^{-1}, ab^{-1} \notin \mathrm{dom}(E)$.
5. If $(a,b), (c,d) \in E$, and $ab^{-1} = (cd^{-1})^{\pm 1}$, then $(a,b) = (c,d)$ or $(a,b) = (d,c)$.
6. If $(a,b), (b,a) \in E$, then $a^{-1} \in \mathrm{dom}(E)$ if and only if $b^{-1} \in \mathrm{dom}(E)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a,b) \in E$ for some $b \in G$.

If $a^{-1} \in \mathrm{dom}(E)$, then by condition (3) we are already done. So reduce to the case when $a^{-1} \notin \mathrm{dom}(E)$. By condition (4), $ab^{-1}, ba^{-1} \notin \mathrm{dom}(E)$. By condition (2), $ab^{-1} \neq 1$.

There are two main subcases to consider.

Case 1a: $(b,a) \in E$.

Note that $b^{-1} \notin \mathrm{dom}(E)$ by condition (6). Take c, d, e, f, g, h, i, j to be independent generators of G (sorry for the re-use of f and h) not appearing in the normal forms of the entries in E, and fix

$$\begin{aligned} E' \;:=\; & E \cup \{(a^{-1}, c), (ca, d), (d, b), (ba^{-1}, e), (e, c), (b^{-1}, f), (fb, g), (g, a), (ab^{-1}, h), (h, f), \\ & (hba^{-1}, i), (i, e), (eab^{-1}, j), (j, h)\}. \end{aligned}$$

Conditions (1) and (2) are clear. Condition (3) is also satisfied, using the fact that the new letters are independent and that $ab^{-1} \neq 1$.

For condition (4), first note that if $(x, y) \in E$ with $x^{-1} \notin \mathrm{dom}(E)$, then $xy^{-1}, yx^{-1} \notin \mathrm{dom}(E)$ since E satisfies condition (4). Thus, the only way that at least one of them is in $\mathrm{dom}(E')$ is if one of them equals ab^{-1} or ba^{-1}. By (5), then $(x,y) = (a,b)$ or $(x,y) = (b,a)$, which is okay. For the new pairs in E', condition (4) is checked directly (using $a, b, ab^{-1}, ba^{-1} \neq 1$).

Condition (5) for E' is quickly verified, by directly checking the five new pairs.

Case 1b: $(b,a) \notin E$.

Now just take c, d, e to be independent generators of G not appearing in the normal forms of the entries in E, and fix

$$E' \;:=\; E \cup \{(a^{-1}, c), (ca, d), (d, b), (ba^{-1}, e), (e, c)\}.$$

All the conditions are verified essentially like before.

Case 2: Assume $a \notin \mathrm{dom}(E)$. If $a^{-1} \in \mathrm{dom}(E)$, then we can repeat Case 1 for a^{-1}, and are done (since it makes the functional equation hold for a^{-1} and its inverse).

So, assume $a, a^{-1} \notin \mathrm{dom}(E)$. If there is a pair $(x, y) \in E$ with $xy^{-1} = a^{\pm 1}$, then apply Case 1 to x, and we reduce to a previous work. So, we may as well assume there is not such pair (x, y). Then by passing to $E \cup \{(a, b)\}$, with b a generator of G not appearing anywhere in E or a, this still satisfies all the conditions and we again reduce to a previous case (and note that we don't break condition (4) because of the previous sentence). $\square$

The functional equation for 3456 is $f(1) = f(f(f(1)^{-1})f(1))$. Using the seed

$$\{(1, x_1), (x_1, x_2), (x_2, x_1), (x_1^{-1}, x_3), (x_3x_1, x_4), (x_4, x_2), (x_2x_1^{-1}, x_5), (x_5, x_3), (x_2^{-1}, x_6),$$
$$(x_6x_2, x_7), (x_7, x_1), (x_1x_2^{-1}, x_8), (x_8, x_6), (x_8x_2x_1^{-1}, x_9), (x_9, x_5), (x_5x_1x_2^{-1}, x_{10}), (x_{10}, x_8)\}$$

and extending to a full function using the previous lemma, we are done.