1. Handling Equation 63

Equation 63 is $x = y \diamond (x \diamond (x \diamond y))$. Work with a carrier set G that is the free group on countably many letters $x_1, x_2, \ldots$. (If easier, the free abelian group will also work. I just personally found it easier to work multiplicatively, and so kept track of orders of products.) Setting $y = hx$ and $a \diamond b = f(ba^{-1})a$, where $f\colon G \to G$ is a function, we are led to the functional equation

$$f(f^2(h)h^{-1}) = h^{-1}.$$

Think of this as saying that if $(a,b),(b,c) \in f$, then we must have $(ca^{-1}, a^{-1}) \in f$.

An important initial seed is the following:

$$\begin{aligned} E_0 \;:=\; &\{(1,x_1),(x_1,x_2),(x_2,1),(x_1^{-1},x_1^{-1}),(x_1x_2^{-1},x_2^{-1}),(x_2^{-1},x_3),(x_3x_2x_1^{-1},x_2x_1^{-1}),\\ &(x_3,x_1x_2^{-1}),(x_2^{-1}x_3^{-1},x_3^{-1})\}. \end{aligned}$$

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

(1) E is finite.
(2) E is a function.
(3) $E_0 \subseteq E$.
(4) If $(a,1) \in E$, then $a = x_2$.
(5) If $(a,b),(b,c) \in E$, then $(ca^{-1}, a^{-1}) \in E$.
(6) If $(a,b),(a',b),(a^{-1},d) \in E$, then $da \neq a'$.
(7) If $(a,b),(a',b),(a^{-1},d),(a'^{-1},d') \in E$, and $da = d'a'$, then $a = a'$.
(8) If $(a,b),(a',b),(a^{-1},d),(a'^{-1},d) \in E$, then $a = a'$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a,b) \in E$ for some $b \in G$.

If $b \in \mathrm{dom}(E)$, then by condition (5) we are already done. So reduce to the case when $b \notin \mathrm{dom}(E)$. Let $a_1, a_2, \ldots, a_n$ be the finite list of first coordinates in E that map to b. In particular, note that this means $a_i \neq 1$ and $a_i \neq q^{-1}$ (for each i) by condition (3). (This, plus the closure condition (5), is the reason we need so many pairs in E_0.)

Let S be the subset of indices j where $(a_j^{-1}, d_j) \in E$ for some $d_j \in G$. Note that $d_j \neq 1$, since $a_j \neq x_2^{-1}$, using condition (4). Also note that if $j,k \in S$ are distinct, then $d_j \neq d_k$ by condition (8), and also by condition (7) we know $d_ja_j \neq d_ka_k$.

Fix c to be the least generator of G that does not appear anywhere in the reduced forms for the coordinates in E. We then take

$$E' := E \cup \{(b,c),(ca_i^{-1},a_i^{-1})\}_{i\in[1,n]} \cup \{(d_ja_jc^{-1},a_jc^{-1}\}_{j\in S}.$$

Conditions (1) and (3) for E' are clear. For (2), use $d_ja_j \neq d_ka_k$ when $j,k \in S$ are distinct. Next, (4) holds since none of the new pairs ends in 1 (because $a_i \neq 1$). A finite check also shows that condition (5) holds (here, we need to use the fact that (6) holds for E to get $da_j \neq a_i$ for any $j \in S$ and $i \in [1,n]$).

Next, (6) holds by considering cases, noting that c is a new generator, and again using $da_j \neq a_i$ (so that $a_ic^{-1} \notin \mathrm{dom}(E')$), and similarly for (7) and (8).

Case 2: Assume $a \notin \mathrm{dom}(E)$. If $(y,a) \in E$ for some $y \in G$, then applying Case 1 to y, we reduce to the situation when $a \in \mathrm{dom}(E)$ after all. Thus, we may assume $a \notin \mathrm{im}(E)$. Using $E \cup \{(a,b)\}$, where b is the least generator of G not appearing in the reduced forms for the coefficients of E, nor of a, we reduce to Case 1 again. $\square$

Equation 1692 is $x = (y \diamond x) \diamond ((y \diamond x) \diamond y)$. The corresponding functional equation is

$$hf(h)^{-1} = f^2(f(h)^{-1}).$$

The seed $E_0 \cup \{(x_4, x_5), (x_5^{-1}, x_6), (x_6, x_7), (x_7 x_5, x_5)\}$ is in $\mathscr{E}$ (by a finite check) and fails this new equation. After extending to a full function, we are done.