1. Equation 917, Hardy

I work with the free multiplicative group G on countably many generators as usual. The functional equation is

$$f(f(f(h)h^{-1}f(1)^{-1})f(1)) = h^{-1}.$$

Think of this as saying that if $(1, x_1), (a, b), (ba^{-1}x_1^{-1}, c) \in f$, then we must have $(cx_1, a^{-1}) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

(1) E is finite.
(2) E is a function.
(3) $(1, x_1) \in E$ (where x_1 is a generator).
(4) If $(a, b), (ba^{-1}x_1^{-1}, c) \in E$, then $(cx_1, a^{-1}) \in E$.
(5) If $(a, b) \in E$, then $ab^{-1} \neq x_1$.
(6) If $(a, b), (c, d) \in E$ and $ab^{-1} = cd^{-1}$, then $(a, b) = (c, d)$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. **Case 1**: Assume $(a, b) \in E$ for some $b \in G$.

If $ba^{-1}x_1^{-1} \in \operatorname{dom}(E)$, then by condition (4) we are already done. So reduce to the case when $ba^{-1}x_1^{-1} \notin \operatorname{dom}(E)$. Let c be any generator not appearing anywhere in E, and fix

$$E' := E \cup \{(ba^{-1}x_1^{-1}, c), (cx_1, a^{-1})\}.$$

Conditions (1) and (3) on E' are clear. Condition (2) on E' follows from (6) for E. Condition (4) on E' comes from condition (5) on E. Condition (5) is easy, and so is condition (6).

Case 2: Assume $a \notin \operatorname{dom}(E)$. If there is some $(x, y) \in E$ with $a = yx^{-1}x_1^{-1}$, then use Case 1 to make the functional equation hold for x, and then a will belong to the domain of that extension, and we reduce to Case 1 again.

So, we may assume there is not such pair (x, y). Taking b to be any generator of G not appearing in E or in a, then $E \cup \{(a, b)\} \in \mathscr{E}$ and we again revert to Case 1. □

With the choice that $f(1) = x_1$, the functional equation for 1629 is $f^2(x_1^{-1}) = x_1^{-1}$. The initial seed

$$\{(1, x_1), (x_1^2, 1), (x_1^{-1}, x_2), (x_2, x_3), (x_3x_1, x_1)\} \in \mathscr{E}$$

works to contradict this equation.

The functional equation for 2441 is $f(f(f(x_1^{-1})x_1)^{-1}) = f(f(x_1^{-1})x_1)^{-1}$. The initial seed

$$\{(1, x_1), (x_1^2, 1), (x_1^{-1}, x_2), (x_2x_1, x_3), (x_3^{-1}, x_4)\}$$

works.