1. 1516 DOES NOT IMPLY 1489

This proof is very similar to the one that handled Obelix. I'll thus be somewhat brief. The functional equation for 1516 is

$$f(f^2(h)h^{-1}f(1)^{-1}) = h^{-1}f(1)^{-1}$$

Throughout, we'll take $(1,1)$ for simplicity. Think of this functional equation as saying that if $(a,b),(b,c) \in f$, then we must have $(ca^{-1}, a^{-1}) \in f$.

Define $\mathscr{E}$ as the collection of sets $E \subseteq G^2$ satisfying the following properties.

(1) E is finite.
(2) E is a function.
(3) $(1,1) \in E$.
(4) If $(a,b),(b,c) \in E$, then $(ca^{-1}, a^{-1}) \in E$.
(5) If $(a,b),(a',b),(a^{-1},d),(a'^{-1},d') \in E$ and $da = d'a'$, then $a = a'$.
(6) If $(a,a^{-1}) \in E$, then $a = 1$.
(7) If $(a,b),(a',b),(a^{-1},d) \in E$, then $ad \neq a'$.
(8) If $(a,1) \in E$, then $a = 1$.

Lemma 1.1. *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for a.*

Proof. Do a case analysis as in my other files. The main case is when $(a,b) \in E$ but $b \notin \mathrm{dom}(E)$. In that case $a, b \neq 1$.

Fix the finite list of first coordinates $a_1, a_2, \ldots, a_n$ that map to b in E. Also, fix the set S to consist of the indices j where $(a_j^{-1}, d_j) \in E$ for some $d_j \in G$. Taking c to be a new generator of the (countably-generated free) group G, fix

$$E' := E \cup \{(b,c)\} \cup \{(ca_i^{-1}, a_i^{-1}\}_{i\in[1,n]} \cup \{d_j a_j c^{-1}, a_j c^{-1}\}_{j\in S}.$$

The last four conditions defining $\mathscr{E}$ help make the case analysis work. $\square$

The functional equation for 1489 is $f(f(f(h^{-1})h)f(h)^{-1}) = hf(h)^{-1}$. Taking the seed

$$\{(1,1),(x_1,x_2),(x_1^{-1},x_3),(x_3x_1,x_4),(x_4x_2^{-1},x_5)\} \in \mathscr{E}$$

works to contradict this equation.