## 1. Equation 1518

Our carrier set will be the (nonabelian) free group $G$ on countably many letters $x_1, x_2, \ldots$. The functional equation for 1518 is

$$f(f(f(h)h^{-1})hf(1)^{-1}) = hf(1)^{-1}.$$

Think of this as saying that once $(1, x_1), (a, b), (ba^{-1}, c) \in f$, then $(cax_1^{-1}, ax_1^{-1}) \in f$. A special family built from this rule is

$$E_0 = \{(1, x_1), (x_1, x_2), (x_2x_1^{-1}, x_1^{-1}), (x_1^{-1}, 1), (x_2x_1^{-2}, x_1^{-2})\}$$

Let $\mathscr{E}$ be the collection of subsets $E \subseteq G^2$ subject to the following conditions:

(1) $E$ is finite.
(2) $E$ is a function.
(3) $E_0 \subseteq E$.
(4) If $(a, b), (ba^{-1}, c) \in E$, then $(cax_1^{-1}, ax_1^{-1}) \in E$.
(5) If $(a, b), (c, d) \in E$ and $ba^{-1} = dc^{-1}$, then $a = cx_1^k$ for some $k \in \mathbb{Z}$.
(6) If $(a, b) \in E$, then $ba^{-1} \neq 1$.
(7) If $(a, b) \in E$ and $b = x_1^k$ for some $k \in \mathbb{Z}$, then $a = x_1^{k-1}$.

**Lemma 1.1.** *For any $E \in \mathscr{E}$ and any $a \in G$, there is an extension $E \subseteq E' \in \mathscr{E}$ where the functional equation holds for $a$.*

*Proof.* **Case 1**: Assume $(a, b) \in E$ for some $b \in G$.

If $ba^{-1} \in \mathrm{dom}(E)$, then we are done by property (4). So we may reduce to the case when $ba^{-1} \notin \mathrm{dom}(E)$. In particular, $b$ is not a power of $x_1$, by (3) and (7).

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be the set of pairs in $E$ such that $b_ia_i^{-1} = ba^{-1}$. By condition (5), we have that $a_i = ax_1^{k_i}$ for some integers $k_1, \ldots, k_n$ (and consequently $b_i = bx_1^{k_i}$ also).

Let $c$ be a generator of $G$ that does not appear in the normal form of any coordinate in $E$ and take

$$E' = E \cup \{(ba^{-1}, c)\} \cup \{(ca_ix_1^{-1}, a_ix_1^{-1})\}_{i \in [1,n]}.$$

Conditions (1), (2), (3), (5), (6), and (7) are clear for $E'$. Condition (4) is a case analysis, and uses (6) and (7) for $E$ to prevent needing new pairs in $E'$. (Recall that $b$ is not a power of $x_1$.)

**Case 2**: Assume $a \notin \mathrm{dom}(E)$. If $(x, y) \in E$ with $yx^{-1} = a$, then applying Case 1 to $x$, we get $a \in \mathrm{dom}(E)$, and then reduce to Case 1. Thus, we may assume there is not such pair. Then letting $b$ be any generator of $G$ not in the normal form of any element in $E$ or of $a$, take $E' := E \cup \{(a, b)\}$, and again reduce to Case 1. $\square$

The functional equation for 47 is

$$f^3(1) = 1.$$

Taking $E_0 \cup \{x_2, x_3)\}$ as the initial seed works.

Equation 614's functional equation is

$$f^2(f(x_1^{-1})x_1) = 1.$$

The same initial seed as before works.

The functional equation for 817 is

$$f(x_1^2) = 1.$$

Using $E_0 \cup \{(x_1^2, x_3)\}$ works.

The functional equation for 3862 is
$$x_1 = f(f(x_1)^{-1})f(x_1).$$
Taking $E_0 \cup \{(x_2^{-1}, x_3), (x_3x_2x_1^{-2}, x_2x_1^{-2}), (x_3x_2x_1^{-3}, x_2x_1^{-3})\}$ works.