We can define the differential form of the continuity equation as over a finite control volume, V, in $\mathbb{R}^3$:

$$\frac{dM}{dt} = -\oiint \rho \boldsymbol{v} \cdot d\boldsymbol{S}$$

Where M is the mass of the control volume and is a function of time, $M : \mathbb{R} \rightarrow \mathbb{R}^3$, t is time a real valued variable, ρ is the density of the fluid and is a function of the positional vector of the fluid and time, $\rho : \mathbb{R} \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3?\ (not\ sure\ if\ this\ is\ how\ density\ would\ \ be\ written)$, $\boldsymbol{v}$ is the fluid velocity and is a function of the positional vector and time (same format as ρ?), and **S** is the surface of the control volume.

As a side note, this is how I understand density and velocity:

$$\rho = \rho(\boldsymbol{r}, t)$$

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$$

However, I'm not entirely sure how to right this correctly in Lean.

We can write mass, M, as the triple integral of density of the control volume:

$$hM: M(t) = \iiint \rho(t) dV$$

Since mass and density are a function of time, but the control volume stays constant, we can write the change in mass over time as:

$$hM': \frac{dM(t)}{dt} = \iiint \frac{d\rho(t)}{dt} dV$$

Using the divergence theorem, we can rewrite the right-hand side of our differential continuity equation as a triple integral over the control volume:

$$\oiint \rho \boldsymbol{v} \cdot d\boldsymbol{S} = \iiint \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) dV$$

Plugging this relation along with hM’ into the differential form and rearranging, we can get our final form of:

$$\iiint \left[\frac{d\rho(t)}{dt} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v})\right] dV = 0$$