Let On be the class of ordinals, and let "nc" be a shorthand for non-constant. For polynomials P, Q, say that $P < Q$ iff $[P(\alpha)] < [Q(\alpha)]$ for α larger than all coefficients of both polynomials. It's easy to see that if $[P(\alpha)] < [Q(\alpha)]$ holds for at least one α larger than all coefficients of both polynomials, then it holds for all α larger than all coefficients of both polynomials. This is the well-ordering of polynomials we will use throughout this document. The following lemma is known, but we will prove it anyway for completeness.

Lemma 1. *Let $\alpha \in On$ be a field that is not algebraically closed. Let P be the first nc polynomial with coefficients $< \alpha$ that does not have a root $< \alpha$. Let n be the degree of P. Then:*

(i) $\alpha^{n'}\beta = [\alpha^{n'}\beta]$ *for* $n' < n$ *and* $\beta < \alpha$.

(ii) $P(\alpha) = 0$.

(iii) $[\alpha^n]$ *is the field generated by ordinals* $\leq \alpha$.

Proof. First, we prove (i) by induction on n', and we also prove that (i) implies (ii) along the way. For the base case, (i) clearly holds for $n' = 0$, and for $\beta = n' = 1$. The case $\beta > n' = 1$ is proven the same way as the case $1 < \beta < \alpha$ of the induction step (in the fourth paragraph of this proof).

Suppose (i) holds for $0 < n' < n$ and everything below it. Then $\alpha^{n'+1} = \alpha\alpha^{n'} = mex(S)$ where $S = \{\alpha'\alpha^{n'} + (\alpha + \alpha')\beta : \alpha' < \alpha \wedge \beta < \alpha^{n'}\}$. Every $\beta < \alpha^{n'}$ can be written uniquely as $Q_0(\alpha)$ for a degree $n'-1$ polynomial Q_0 with coefficients $< \alpha$. Suppose $Q_0(x) = x^{n'-1}\beta_{n'-1} + \ldots + x\beta_1 + \beta_0$. Then we have $\alpha'\alpha^{n'} + (\alpha+\alpha')Q_0(\alpha) = Q(\alpha)$, where $Q(x) = x^{n'}(\alpha' + \beta_{n'-1}) + x^{n'-1}(\alpha'\beta_{n'-1} + \beta_{n'-2}) + \ldots + x(\alpha'\beta_1 + \beta_0) + \alpha'\beta_0$. Then to see whether a given $\gamma < [\alpha^{n'+1}]$ is in S, we simply need to decide the existence of $\alpha', \beta_0, \beta_1, \ldots, \beta_{n'-1}$ such that we have $Q(\alpha) = \gamma$. Letting $\gamma = \alpha^{n'}\gamma_{n'} + \ldots + \alpha\gamma_1 + \gamma_0$, we thus get the system of equations $\alpha' + \beta_{n'-1} = \gamma_{n'}$ and $\alpha'\beta_i + \beta_{i-1} = \gamma_i$ for $i < n'$, with $\beta_{-1} = 0$. The first gives us $\beta_{n'-1} = \alpha' + \gamma_{n'}$, and then isolating β_{i-1} in the rest in descending order of i inductively gives us $\beta_{i-1} = R_{\gamma,i}(\alpha')$ where $R_{\gamma,i}(x) = x^{n'-i+1} + x^{n'-i}\gamma_{n'} + \ldots + x\gamma_{i+1} + \gamma_i$. Then applying this to $i = 0$ we get $R_{\gamma,0}(\alpha') = \beta_{-1} = 0$ where $R_{\gamma,0}(x) = x^{n'+1} + x^{n'}\gamma_{n'} + \ldots + x\gamma_1 + \gamma_0$, which means $\gamma \in S$ iff $R_{\gamma,0}$ has a root $\alpha' < \alpha$.

In the case that $n' + 1 = n$, the IH is the full statement of (i), and we've just shown that α^n is the smallest $\gamma < [\alpha^n]$ such that $R_{\gamma,0}(\alpha)$ does not have a root $< \alpha$, if such γ exists. But notice that for $\gamma_1 < \gamma_2$, we have $[R_{\gamma_1,0}(\alpha)] = [\alpha^n + \gamma_1] < [\alpha^n + \gamma_2] = [R_{\gamma_2,0}(\alpha)]$, so $R_{\gamma_1,0} < R_{\gamma_2,0}$, i.e. the ordering of the $R_{\gamma,0}$ polynomials is the same as the ordering of their indices. The leading coefficient of P is 1, because otherwise we could divide P by its leading coefficient and get a smaller polynomial without roots in α. Then $P = R_{\gamma,0}$ for some γ, and since the ordering of the $R_{\gamma,0}$ polynomials is the same as the ordering of their indices, this γ with $P = R_{\gamma,0}$ is in fact the smallest γ such that $R_{\gamma,0}$ does not have a root in α. Then as we've proven, $\alpha^n = \gamma$, so $P(\alpha) = R_{\gamma,0}(\alpha) = \alpha^n + \alpha^{n'}\gamma_{n'} + \ldots + \alpha\gamma_1 + \gamma_0 = \alpha^n + \gamma = \gamma + \gamma = 0$, proving that (i) implies (ii).

In the case that $n'+1<n$, by the minimality of P, every polynomial with degree $<n$ and coefficients in α has a root in α, so in particular every $R_{\gamma,0}$ has a root $\alpha'<\alpha$. Therefore S contains every $\gamma<[\alpha^{n'+1}]$, and clearly that is all it contains since Q is a degree n' polynomial and thus $Q(\alpha)<[\alpha^{n'+1}]$ by the IH. So $\alpha^{n'+1}=mex(S)=mex([\alpha^{n'+1}])=[\alpha^{n'+1}]$. Now we only need to prove that $\alpha^{n'+1}\beta=[\alpha^{n'+1}\beta]$ for $1<\beta<\alpha$, which is done by induction on β. Suppose it holds for all $\beta'<\beta$. Then $\alpha^{n'+1}\beta=mex(S_\beta)$ where $S_\beta=\{\gamma(\beta+\beta')+\alpha^{n'+1}\beta' : \beta'<\beta\wedge\gamma<\alpha^{n'+1}\}$. Of course, every $\gamma<\alpha^{n'+1}$ is of the form $\alpha^{n'}\gamma_{n'}+\ldots+\alpha\gamma_1+\gamma_0$ for $\gamma_i<\alpha$. Then to see that $\delta<[\alpha^{n'+1}\beta]$ is in S_β, we only need to write δ as $\alpha^{n'+1}\beta'+\alpha^{n'}\delta_{n'}+\ldots+\alpha\delta_1+\delta_0$, and then we can choose $\gamma_i=\frac{\delta_i}{\beta+\beta'}$ ($\beta'<\beta$ so the denominator is nonzero) and we get $\delta=\gamma(\beta+\beta')+\alpha^{n'+1}\beta'\in S_\beta$. Therefore S_β contains every $\delta<[\alpha^{n'+1}\beta]$, and clearly that is all it contains - letting $\xi=\gamma(\beta+\beta')<\alpha^{n'+1}$ we have $\xi+\alpha^{n'+1}\beta'=[\alpha^{n'+1}\beta'+\xi]<[\alpha^{n'+1}\beta]$). So $\alpha^{n'+1}\beta=mex(S_\beta)=mex([\alpha^{n'+1}\beta])=[\alpha^{n'+1}\beta]$. This proves (i) for n'+1, which completes the induction.

So we have proven (i) and (ii). Now all that is left is proving (iii). From (i) it is clear that the ordinals $\leq\alpha$ do indeed generate everything $<[\alpha^n]$, so we only need to prove that $[\alpha^n]$ is closed under addition, multiplication and inverses. Closure under addition is trivial, and closure under multiplication follows easily from $\alpha^n<[\alpha^n]$ and (i). We prove the existence of inverses once again by induction on $n'<n$. The inductive hypothesis is that everything below $\alpha^{n'}$ has an inverse below $[\alpha^n]$, which is trivial for $n'=1$. Suppose that $\alpha^{n'}\leq\beta<[\alpha^{n'+1}]$ with $n'<n$. Let $\beta=\alpha^{n'}\beta_{n'}+\ldots+\alpha\beta_1+\beta_0$ and let $\alpha^n=\alpha^{n-1}\gamma_{n-1}+\ldots+\alpha\gamma_1+\gamma_0$ with $\gamma_i<\alpha$. We will find a nonzero $\delta<[\alpha^{n-n'+1}]$ with $\beta\delta<\alpha^{n'}$. Let $\delta_{n-n'}<\alpha$ be positive. Suppose we have chosen $\delta_{n-n'},\ldots,\delta_{k+1}$ so that $\beta(\alpha^{n-n'}\delta_{n-n'}+\ldots+\alpha^{k+1}\delta_{k+1})<[\alpha^{n'+k+1}]$ has ξ as the coefficient of $\alpha^{n'+k}$ when written as a degree $n'+k$ polynomial in α with coefficients $<\alpha$. Let $\delta_k=\frac{\xi}{\beta_{n'}}$. Then we have $\beta(\alpha^{n-n'}\delta_{n-n'}+\ldots+\alpha^{k+1}\delta_{k+1}+\alpha^k\delta_k)<\alpha^{n'+k}$. After repeating this until we choose $\delta_{n-n'},\ldots,\delta_1,\delta_0$, let $\delta=\alpha^{n-n'}\delta_{n-n'}+\ldots+\alpha\delta_1+\delta_0$. Then by our choice of δ_0, we have $\beta\delta<\alpha^{n'}$, so $\beta\delta$ has an inverse by IH, and then $\frac{1}{\beta}=\delta\frac{1}{\beta\delta}<[\alpha^n]$ because $[\alpha^n]$ is closed under multiplication. Therefore every $\alpha^{n'}\leq\beta<[\alpha^{n'+1}]$ has an inverse below $[\alpha^n]$, and together with the IH, it shows that everything below $[\alpha^{n'+1}]$ has an inverse below $[\alpha^n]$. In the case $n'+1<n$, this completes the induction step, so by induction this is true for all $n'<n$, and in the case $n'+1=n$, it means that $[\alpha^n]$ is closed under multiplicative inverses, finishing the proof of (iii). □

For natural numbers i, let $0_i = 0$ (this is only so that we can keep track of indices of inputs even when there's an arbitrarily long run of zeroes).

For natural n and ordinals $\eta_1, \eta_2, \ldots, \eta_n$, denote by $A(\eta_n, \ldots, \eta_2, \eta_1)$ the class of all ordinals of the form $\varphi(\eta_n, \ldots, \eta_2, \eta_1, \beta)$ where $\beta \in On$.

For ordinals α and polynomials P, let $C(\alpha, P)$ be the closure of α under minimal roots of nc polynomials $< P$, in the sense that $C(\alpha, P)$ is the smallest field $S \supseteq \alpha$ such that for all nc polynomials $Q < P$ with coefficients in S, the smallest root of Q is in S.

Note that $C(\alpha, P)$ is an ordinal, because we can start from α and iterate extending by the root of the smallest nc polynomial that does not have a root until this polynomial would be $\geq P$, and by Lemma 1, at all steps of this process we will have an ordinal. Also, $C(\alpha, P)$ is in fact closed under all roots of nc polynomials $< P$, because if $Q < P$ has coefficients in $C(\alpha, P)$, then letting $r_1, r_2, \ldots, r_k$ be all roots of Q in $C(\alpha, P)$ and letting $Q'(x) = \frac{Q(x)}{(x+r_1)(x+r_2)\ldots(x+r_k)}$, we have $Q' < Q < P$, which means either Q' has roots in $C(\alpha, P)$ or Q' is constant. The former contradicts $r_1, r_2, \ldots, r_k$ being *all* roots of Q in $C(\alpha, P)$, so Q' is constant, and therefore $r_1, r_2, \ldots, r_k$ are all the roots of Q in On_2.

Lemma 2. *Let $P(x) = x^n\eta_n + \ldots + x^2\eta_2 + x\eta_1$ be a degree n polynomial with constant coefficient 0. Let $\alpha \in On$ be a field that contains all coefficients of P. Then $C(\alpha, P)$ is at most the smallest element of $A(\eta_n, \ldots, \eta_2, \eta_1)$ above α.*

Proof. By induction on P. Since α is required to be a field, this is trivial for linear P, so we can assume $n \geq 2$. Suppose that it holds for all $P' < P$. Let m be minimal such that $\eta_{m+1} > 0$.

If η_{m+1} is a limit ordinal, then for every nc $Q < P$ we have $Q < P'$ for some $P'(x) = x^n\eta_n + \ldots + x^{m+2}\eta_{m+2} + x^{m+1}\eta'_{m+1}$ with $\eta'_{m+1} < \eta_{m+1}$, thus $C(\alpha, P)$ is the union of $C(\alpha, P')$ for $P' < P$. By the IH, $C(\alpha, P')$ is at most the smallest element of $A(\eta_n, \ldots, \eta_{m+2}, \eta'_{m+1}, 0_m, \ldots, 0_2, 0_1)$ above α, which is at most the smallest element of $A(\eta_n, \ldots, \eta_{m+2}, \eta_{m+1}, 0_m, \ldots, 0_2, 0_1) = A(\eta_n, \ldots, \eta_2, \eta_1)$ above α, and therefore $C(\alpha, P)$ is also at most that element.

If $m = 0$ and $\eta_1 = \eta'_1 + 1$ is a successor ordinal, then let $P_\gamma(x) = x^n\eta_n + \ldots + x^2\eta_2 + x\eta'_1 + \gamma$ for ordinals γ. Let $\alpha_0 = C(\alpha, P_0)$, for ordinals γ let $\alpha_{\gamma+1} = C(\alpha_\gamma, P_{\gamma+1})$, and for limit ordinals γ let $\alpha_\gamma = \sup_{\gamma' < \gamma} \alpha_{\gamma'}$. Let $f(\beta) = \varphi(\eta_n, \ldots, \eta_2, \eta'_1, \beta)$ so that f enumerates the elements of $A(\eta_n, \ldots, \eta_2, \eta'_1)$, and let $f(\beta_0)$ be the smallest element of $A(\eta_n, \ldots, \eta_2, \eta'_1)$ above α. We will prove by induction on γ that $\alpha_\gamma \leq f(\beta_0 + \gamma)$. To disambiguate between the two inductive hypotheses, we will refer to the inductive hypothesis for the main statement as "outer IH", and to the IH for this statement as "inner IH". For $\gamma = 0$, we have $\alpha_0 \leq f(\beta_0) = f(\beta_0 + 0)$ from the outer IH. For $\gamma = \gamma' + 1$, $\alpha_{\gamma'}$ by definition contains the roots of all polynomials $< P_{\gamma'}$ with coefficients from $\alpha_{\gamma'}$. Therefore if it contains a root of $P_{\gamma'}$, then $\alpha_\gamma = C(\alpha_{\gamma'}, P_{\gamma'+1}) = \alpha_{\gamma'} \leq f(\beta_0 + \gamma') < f(\beta_0 + \gamma)$. If $\alpha_{\gamma'}$ does not contain a root of $P_{\gamma'}$, then by Lemma 1(ii), we have $P_{\gamma'}(\alpha_{\gamma'}) = 0$, and by Lemma 1(iii) and $n \geq 2$, $[\alpha_{\gamma'}^n]$ is a field containing this root of $P_{\gamma'}$. Then it contains roots of all $P_{\gamma''}$ with $\gamma'' \leq \gamma'$, which are all the polynomials $P_0 \leq Q < P_{\gamma'+1}$, so $\alpha_\gamma = C(\alpha_{\gamma'}, P_{\gamma'+1}) = C([\alpha^n_{\gamma'}], P_0)$. By the outer IH, that is at most the smallest element of $A(\eta_n, \ldots, \eta_2, \eta'_1)$ above $[\alpha^n_{\gamma'}]$, and since all elements of this class are epsilons and there are no epsilons between $\alpha_{\gamma'}$ and $[\alpha^n_{\gamma'}]$, we see that α_γ is at most the smallest element of $A(\eta_n, \ldots, \eta_2, \eta'_1)$ above $\alpha_{\gamma'}$. Then inner IH gives us $\alpha_{\gamma'} \leq f(\beta_0 + \gamma')$, so because f enumerates the elements of $A(\eta_n, \ldots, \eta_2, \eta'_1)$, we get $\alpha_\gamma \leq f(\beta_0 + \gamma)$. Finally, for limit γ, we have $\alpha_\gamma = \sup_{\gamma'<\gamma} \alpha_{\gamma'}$ and $f(\beta_0 + \gamma) = \sup_{\gamma'<\gamma} f(\beta_0 + \gamma')$, so from inner IH we get $\alpha_\gamma \leq f(\beta_0 + \gamma)$. This completes the inner induction. Now let δ be the smallest element of $A(\eta_n, \ldots, \eta_2, \eta_1)$ above α. By definition of φ, δ is closed under f, and we clearly also have $\beta_0 \leq \alpha + 1 < \delta$, so δ is closed under $\gamma \mapsto \beta_0 + \gamma$, meaning that δ is closed under $\gamma \mapsto f(\beta_0 + \gamma)$, and by $\alpha_\gamma \leq f(\beta_0 + \gamma)$ it is also closed under $\gamma \mapsto \alpha_\gamma$. Then take any nc polynomial $Q < P$ with coefficients from δ. Let $\gamma < \delta$ be larger than all coefficients of Q. Then $Q < P_\gamma$, so $\alpha_\gamma < \delta$ contains a root of Q, and so does δ. Therefore $\delta > \alpha$ is closed under roots of polynomials $< P$, which means $C(\alpha, P)$ is at most δ.

If $m > 0$ and $\eta_{m+1} = \eta'_{m+1} + 1$ is a successor ordinal, then let $\alpha_0 = \alpha$, and for natural k, let α_{k+1} be the union of $C(\alpha_k, P_\gamma)$ for all $P_\gamma(x) = x^n\eta_n + ... + x^{m+2}\eta_{m+2} + x^{m+1}\eta'_{m+1} + x^m\gamma$ with $\gamma < \alpha_k$. Let α_ω be the union of α_k for natural k. Each $C(\alpha_k, P_\gamma)$ is at most the smallest element of $A(\eta_n, \ldots, \eta_{m+2}, \eta'_{m+1}, \gamma, 0_{m-1}, ..., 0_2, 0_1)$ above α_k, which is less than the ordinal $\varphi(\eta_n, \ldots, \eta_{m+2}, \eta'_{m+1}, \alpha_k, 0_{m-1}, ..., 0_1, 1)$, therefore we have that α_{k+1} is at most $\varphi(\eta_n, \ldots, \eta_{m+2}, \eta'_{m+1}, \alpha_k, 0_{m-1}, ..., 0_1, 1)$. By definition of φ, the elements of $A(\eta_n, \ldots, \eta_{m+2}, \eta_{m+1}, 0_m, 0_{m-1}, ..., 0_2, 0_1) = A(\eta_n, \ldots, \eta_2, \eta_1)$ are closed under $\beta \mapsto \varphi(\eta_n, \ldots, \eta_{m+2}, \eta'_{m+1}, \beta, 0_{m-1}, ..., 0_1, 1)$, so by induction on k, all α_k are less than the smallest element of $A(\eta_n, \ldots, \eta_2, \eta_1)$ above α_0, and thus their union α_ω is at most that element. We only need to show that $C(\alpha, P)$ is at most α_ω. Take any nc $Q < P$ with coefficients from α_ω. Let δ be the coefficient of x^m in $Q(x)$, and let k be large enough that all coefficients of $Q(x)$ are $< \alpha_k$. Then for $\delta < \gamma < \alpha_k$, we have $Q < P_\gamma$, so $C(\alpha_k, P_\gamma)$ contains a root of Q, which means α_{k+1} contains that root of Q, and so does α_ω. Therefore α_ω is closed under roots of polynomials $< P$, which means $C(\alpha, P) \leq \alpha_\omega$. □

Let $A(1@\omega)$ be the intersection of $A(1, 0_n, \ldots, 0_2, 0_1)$ for natural n, and let $\beta \mapsto \varphi(1@\omega, \beta)$ enumerate the elements of $A(1@\omega)$.

Corollary 3. *Every ordinal of the form $\varphi(1@\omega, \beta)$ with $\beta \in On$ is an algebraically closed field of nimbers.*

Proof. Let $\tau = \varphi(1@\omega, \beta)$, and let $\alpha < \tau$ be a field that is not algebraically closed. Then for natural n, letting $P_n(x) = x^{n+1}$, we see that the closure $C(\alpha, P_n)$ of α under roots of polynomials of degree $\leq n$ is at most the smallest element of $A(1, 0_n, \ldots, 0_2, 0_1)$ above α, which is less than τ. Therefore the algebraic closure of α, which is the union of $C(\alpha, P_n)$ for all natural n, is at most the smallest element of $A(1@\omega)$ above α, which is at most τ.

If there is a field $\alpha < \tau$ such that the algebraic closure of α is τ, then τ is algebraically closed. Otherwise, since for every β, there is a field $\beta < \alpha \leq [\beta^\beta]$, and unions of $\subseteq$-chains of fields are also fields, this means that there are arbitrarily large fields $\alpha < \tau$, and their algebraic closures are therefore arbitrarily large algebraically closed fields $\tau' < \tau$, which makes τ itself an algebraically closed field. $\square$