Generalised Banach Steinhaus - Notes

THEOREM 1 (BANACH–STEINHAUS–MACKEY) *Let $E$ be a Hausdorff* LCTVS. *Every bounded complete disk of $E$ is strongly bounded.*

In other words:

COROLLARY 1 *Let $E$ and $F$ be* LCTVS, *$E$ Hausdorff; then every subset of $L(E, F)$ bounded for pointwise convergence is bounded for the uniform convergence on the bounded complete disks of $E$.*

*Proof of Theorem* 1. We must show that if $M$ is a set of continuous linear mappings from $E$ into $F$, bounded for pointwise convergence, and if $A$ is a complete bounded disk in $E$, then $M(A)$ is a bounded subset of $F$. Now, let $E_A$ be the vector space generated by $A$, with the *gauge* semi-norm of $A$:

$$\| x \|_A = \inf_{x \in \lambda A} | \lambda |;$$

this is a norm since $A$ is bounded. Taking the restrictions of $u \in M$ to $E_A$, we can show that the set of mappings from $E_A$ into $F$ thus obtained, is bounded for $A$-convergence. But as this set is bounded for pointwise convergence, it suffices to show that $E_A$ is complete and to apply the Banach–Steinhaus theorem (Chapter 1, Section 15, Theorem 11). We then have the following lemma, interesting in itself:

LEMMA 1 *Let $E$ be a Hausdorff* LCTVS, *$A$ a complete bounded disk in $E$. Then the corresponding normed space $E_A$ is a Banach space, i.e. it is complete.*

*Proof* As $A$ is closed in $E$, and therefore contains the ends of the intervals intersected by $A$ on the real lines passing through the origin, we conclude that the unit ball of $E_A$ is $A$. Also it is clear that a normed space is complete if and only if its unit ball is complete. It is then sufficient to show that the unit ball $A$ of $E$ is complete for the norm topology of $E_A$. This follows from Chapter 2, Section 18, Proposition 35, applied to $E_A$ and to the topology induced by $E$ on $E_A$.

Required concepts: (the ones marked with ? I wasn’t able to find in mathlib)

- -> locally convex topological vector spaces
- -> Hausdorff spaces (i.e. two distinct points => exist two disjoint neighbourhoods)
- -> ? bounded complete disks
- -> ? vector space generated by A
- -> gauge seminorm

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```
@[class]
structure locally_convex_space (𝕜 : Type u_1) (E : Type u_2) [ordered_semiring 𝕜]
    [add_comm_monoid E] [module 𝕜 E] [topological_space E] :
  Prop
(convex_basis : ∀ (x : E), (nhds x).has_basis (λ (s : set E), s ∈ nhds x ∧ convex 𝕜 s) id)
```

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A `locally_convex_space` is a topological semimodule over an ordered semiring in which convex neighborhoods of a poi form a neighborhood basis at that point.

▶ Instances of this typeclass

```
@[class]
structure is_Hausdorff {R : Type u_1} [comm_ring R] (I : ideal R) (M : Type u_2)
    [add_comm_group M] [module R M] :
  Prop
(haus' : ∀ (x : M), (∀ (n : ℕ), x ≡ 0 [SMOD I ^ n • ⊤]) → x = 0)
```

source

A module `M` is Hausdorff with respect to an ideal `I` if `⋂ I^n M = 0`.

▶ Instances of this typeclass

```
noncomputable def gauge_seminorm {𝕜 : Type u_1} {E : Type u_2} [add_comm_group E]
    [module ℝ E] {s : set E} [is_R_or_C 𝕜] [module 𝕜 E] [is_scalar_tower ℝ 𝕜 E]
    (hs₀ : balanced 𝕜 s) (hs₁ : convex ℝ s) (hs₂ : absorbent ℝ s) :
  seminorm 𝕜 E
```

source

`gauge s` as a seminorm when `s` is balanced, convex and absorbent.

▶ Equations