Smooth partition of unity and the weak Whitney embedding theorem

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Abstract

The weak Whitney embedding theorem says that a σ-compact smooth m-dimensional manifold can be embedded into $\mathbb{R}^{2m+1}$. This paper describes the current state of a project aimed to formalize this theorem in the Lean theorem prover. Currently, I formalized a "baby" version of this theorem and the existence of a smooth partition of unity subordinate to a given open covering.

1 Introduction

Recall that a *smooth partition of unity* on a smooth manifold M is a collection of smooth functions $\{ f_i : M \to \mathbb{R} \}_{i \in \mathcal{I}}$ such that

- each f_i takes values on $[0, 1]$;
- the supports of f_i form a locally finite family of sets, i.e., for every point x on M there exists a neighbourhood $U \ni x$ such that all but finitely many f_i vanish on U;
- the sum $\sum_i f_i(x)$ equals 1 at every point x; this sum makes sense due to the previous assumption.

If M is a σ-compact Hausdorff topological space[1], then for every covering $\{ U_i \}_{i \in \mathcal{I}}$ of M by open sets there exists a partition of unity $\{ f_i \}_{i \in \mathcal{I}}$ that is subordinate to this covering, i.e., for each i, U_i includes the topological support of f_i.

Smooth partitions of unity are used in many definitions and theorems about smooth manifolds, including, e.g., the definition of the integral of a differential form over a manifold, the construction of the current defined by a smooth foliation with a transverse invariant measure, and the proof of the weak Whitney embedding theorem.

More examples

`Mathlib` is a project aimed to formalize lots of real-world mathematics in the Lean proof assistant (currently we use a community fork of Lean 3; we are working on migration to Lean 4). To the best of my knowledge, Lean is the only proof assistant with a formalization of manifolds, and my project is the only formalization of smooth partitions of unity available for Lean, hence it is the only formalization of smooth partitions of unity.

In: A. Editor, B. Coeditor (eds.): Proceedings of the XYZ Workshop, Location, Country, DD-MMM-YYYY, published at http://ceur-ws.org

[1] In many books one or both of these assumptions are a part of the definition of a manifold. Mathlib's definiton does not include these assumptions because many theorems still hold true without them.

2 Implementation details

2.1 Ingredients of the proof

In this section I briefly describe notions and theorems formalized as a part of the project.

2.1.1 Manifolds

During the past two years, Sébastien Gouëzel formalized the notion of an infinitely smooth manifold in Lean. This is not my work and I hope that some day Sébastien will write a paper about this formalization, so I will only list a few design choices that are important for my project. Understanding these choices may help those who will decide to read the code.

First, a `charted_space` is a topological space M with an atlas of charts taking values in another topological space H. We also assume that there is a choice of the "canonical" chart at each point $x \in M$, denoted by `chart_at H x`.

While the definition works in much more general cases, the main two examples are $H = \mathbb{R}^n$ and $H = \{ x \in \mathbb{R}^n \mid 0 \le x_1 \}$. We do not assume that H is a subset of $\mathbb{R}^n$ so that H can be the space itself, not the set of all points of the space. Instead, we fix a closed embedding of $I \colon H \to E$ into a normed vector space E with a left inverse $I^{-1} \colon E \to H$ that is continuous on the whole space. This embedding is called a *model with corners*. Then we use I to turn charts `chart_at` into *extended charts* taking values in E and define smoothness etc. in terms of these charts. Most parts of the proofs about smooth bump functions are done in the extended charts.

2.1.2 Paracompactness

A topological space is said to be *paracompact* if every open covering admits a locally finite refinement. For a finite dimensional Hausdorff topological manifold, paracompactness is equivalent to σ-compactness of each connected component. I did not need and have not formalized the forward implication.

The reverse implication follows from a more general fact: a locally compact σ-compact Hausdorff topological space is a paracompact space. I prove a more precise version of this statement: if X is a locally compact σ-compact Hausdorff topological space and for each point x in X, $\{ B_\alpha \}_{\alpha \in \mathcal{I}(x)}$ is a basis of the neighbourhoods filter $\mathcal{N}(x)$, then each open covering of X admits a locally finite refinement that consists of elements of the bases B_α. I use this lemma to find a locally finite covering of the manifold by supports of smooth bump functions. One can use the same lemma to find, e.g., a locally finite refinement by open balls in a metric space.

While I do not need it for smooth partitions of unity, I also formalize Mary Rudin's proof [?] of the fact that every (extended) metric space is paracompact.

2.1.3 Shrinking lemma

Shrinking lemma says that a point finite open covering $\{ U_i \}_{i \in \mathcal{I}}$ of a normal topological space admits an open refinement $\{ V_i \}_{i \in \mathcal{I}}$ such that the closure of each V_i is included by U_i. I formalize this lemma for general topological spaces, as well as provide specialized versions for (extended) metric spaces.

2.1.4 Urysohn's Lemma

Urysohn's lemma says that for any two disjoint closed sets in a normal topological space, there exists a continuous function that is equal to zero on the first set and is equal to one on the second set. While this lemma is not needed for construction of a smooth partition of unity, only for its continuous counterpart, I formalize it (and continuous partition of unity).

Some notes about my formalization of Urysohn's lemma?

2.1.5 Smooth bump functions

Given a normed vector space V and a point $c \colon V$, a *smooth bump function* centered at c is an infinitely smooth function $f \colon V \to \mathbb{R}$, $0 \le f(x) \le 1$, that is equal to one in a neighborhood of $c \colon V$ and is equal to zero outside of an open ball centered at c.

The construction of a smooth bump function starts with the function

$$f_1(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

This is an infinitely smooth function that is equal to zero on $(-\infty, 0]$ and is positive on $(0, +\infty)$. These properties were formalized by Sébastien Gouëzel in 2020.

Next, the infinitely smooth function $f_2(x) = \frac{f_1(x)}{f_1(x)+f_1(1-x)}$ equals zero on $(-\infty, 0]$, equals one on $[1, +\infty)$, and takes values between zero and one on $(0, 1)$. This function is formalized as `real.smooth_transition`.

If E is an inner product space, then one can define an infinitely smooth bump function using the formula $f_{3,c}(x) = \frac{R-\|x-c\|}{R-r}$. This function equals one on the closed ball $\overline{B}(c, r)$ and vanishes outside of the open ball $B(c, R)$. This construction works in any inner product space, not only finite dimensional spaces.

If V is a finite dimensional real vector space, then there exists a linear equivalence e between V and the standard Euclidean space E of the same dimension. The function $f_4(x) = f_{3,e(c)}(e(x))$ is an infinitely smooth bump function. It is equal to one on $e^{-1}(\overline{B}(e(c), r))$ and vanishes outside of $e^{-1}(B(e(c), R))$. I call these preimages *euclidean balls*, see Sec. 2.1.6 for details. This construction works only in a finite dimensional real vector space but does not require that the norm comes from an inner product space structure.

Finally, if c is a point on a finite dimensional manifold M, $\xi\colon U \to V$, $c \in U \subset V$ is a smooth chart, and R is so small that the range of ξ includes the closed euclidean ball $\overline{B}_{eu}(\xi(c), R)$, then $f_4(\xi(c)) \circ \xi$ is a smooth bump function on M. The actual definition is a bit more convoluted to make it work for manifolds with boundary.

2.1.6 Unspecified Euclidean space structure

When we construct a smooth bump function, we first define it on an inner product space, then use the fact that any finite dimensional real vector space is isomorphic to the Euclidean space of the same dimension to transfer the smooth bump function to any finite dimensional real vector space.

Many properties of the resulting function are best formulated in terms of the distance transferred from the Euclidean space, not in terms of the original distance on a finite dimensional vector space.

So, I define `euclidean.dist` to be this distance and build a minimalistic API about this definition. This way proofs about smooth bump functions on a finite dimensional real vector space V do not need to explicitly use an equivalence between V and the Euclidean space of the same dimension.

2.1.7 Bump function covering

A collection of infinitely smooth functions $\{ f_i \}_{i\in\mathcal{I}}$ is a *covering by supports of smooth bump functions* if the following conditions hold true:

- $0 \le f_i(x) \le 1$ for all i and x;
- supports of f_i is a locally finite family of sets;
- for each x there exists i such that f_i equals one in a neighborhood of x.

The formal definition assumes that f_i are smooth bump functions described in Sec. 2.1.5.

The main fact about these coverings says that every family of neighborhoods $\{ U_x \}_{x\in M}$, $U_x \in \mathcal{N}(x)$, admits a covering $\{ f_i \}_{i\in\mathcal{I}}$ by supports of smooth bump functions such that $\overline{\operatorname{supp}(f_i)} \subset U_{c_i}$, where c_i is the center of f_i. This fact almost immediately follows from existence of smooth bump functions, the shrinking lemma, and a version of the lemma about paracompactness of a locally compact σ-compact space, see Sec. 2.1.2.

The main reason to deal with coverings of this type is to use them to define a partition of unity, see below. I also directly use a covering by supports of smooth bump functions to prove a simple particular case of the weak Whitney embedding theorem: a compact manifold can be embedded into $\mathbb{R}^n$ for some sufficiently large n.

2.1.8 Partition of unity

A *smooth partition of unity* is a collection of smooth nonnegative functions $\{ g_i\colon M \to \mathbb{R} \}_{i\in\mathcal{I}}$ such that supports $\operatorname{supp} g_i$ form a locally finite collection of sets and for each x we have $\sum_i g_i(x) = 1$.

If $\{ f_i \}_{i\in\mathcal{I}}$ is a covering by supports of smooth bump functions, then the functions g_i given by

$$g_i(x) = f_i(x) \prod_{j<i} (1 - f_j(x))$$

form a smooth partition of unity. Thus every open covering $\{ U_i \}$ admits a subordinate partition of unity.

2.1.9 Hausdorff dimension

All proofs of the weak Whitney embedding theorem I know about rely on the following particular of the Sard's Theorem: if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a smooth (or, more generally, lcally Lipschitz continuous) map and $n < m$, then the range of f has measure zero, hence its complement is everywhere dense.

The most straightforward way to prove this fact is to say that the Hausdorff dimension of the range is at most $\dim_H(\mathbb{R}^n) = n < m = \dim_H(\mathbb{R}^m)$. I have formalized the definition of the Hausdorff dimension of a metric space. In order to prove this theorem, I still need to formalize $\dim_H(\mathbb{R}^n) = n$ and the fact that a locally Lipschitz continuous map does not increase the Hausdorff dimension.

2.2 Design choices

2.2.1 Covering of subsets

Most sources (including the rest of this paper) define partition of unity etc. only for coverings of the whole space. At the same time, quite a few proofs need only a part of the covering that covers a specified closed set. While in traditional proofs it is easy to jump between open coverings of a closed set and open coverings of the whole space, it is much less convenient to do in formal proofs. So, I first prove most lemmas for a covering of a closed set s, then apply them to $s = univ$[2].

2.2.2 Sums with finitely many non-zero terms

When I started working on this project, Mathlib had two functions formalizing the notion of a finite sum: `finset.sum` and `finsupp.sum`. The former function takes a finite set `s : finset` α and a function $f\colon \alpha \to M$, where M is an additive commutative monoid. The latter function takes a function $f\colon \alpha \to M$ such that the set $\operatorname{supp} f = \{\, x \mid f(x) \neq 0 \,\}$ is finite[3], and another function $g\colon \alpha \to M \to N$, then computes $\sum_{x \in \operatorname{supp} f} g(x, f(x))$; this sum has nice properties provided that g is additive in its second argument.

One of the assumptions in the definition of a partition of unity is that for each x, the sum of all nonzero $f_i(x)$ equals one. While it is possible to express this using `finset.sum`, it would be not convenient to deal with this definition for a few reasons.

- one needs a proof of finiteness of $\{\, i \mid f_i(x) \neq 0 \,\}$ to *state* the property $\sum_i f_i(x) = 1$, making the definition of the structure and basic lemmas about it much less readable;
- one would need a separate lemma stating that actually one can use any other finite set s such that $f_i(x) = 0$ for all $i \notin s$ instead of $\{\, i \mid f_i(x) \neq 0 \,\}$;
- the same difficulties appear with other sums that involve $f_i(x)$.

While the `finsupp.sum` API solves the last two issues, it still requires a proof of finiteness to state the property. Also, the standard proof of existence of a partition of unity subordinate to a given open covering involves products with finitely many non-one multipliers, and `finsupp.sum` has no multiplicative counterpart.

Fortunately, at about the same time Kexing Ying and Kevin Buzzard submitted a draft of one more function formalizing a sum of finitely many terms. The new function `finsum` takes a function $f\colon \alpha \to M$, where M is an additive commutative monoid, and returns $\sum_i f(i)$ if f vanishes at all but finitely many points, and zero otherwise. This approach makes all the problems mentioned above go away (more precisely, to API lemmas about the new function). I added a multiplicative version `finprod` and polished the API, then used it in my project.

Now the assumption looks like $\sum^f i$`, f i x = 1`

3 Future plans

The main goal of this project is to formalize the weak Whitney embedding theorem. The main missing ingredient is a particular case of Sard's Theorem: if $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map and $m < n$, then the complement to the range of f is everywhere dense in the codomain. This lemma easily follows from three helper lemmas:

- the Hausdorff dimension of $\mathbb{R}^n$ equals n;

[2] In Lean, `univ` is the set that contains all elements of a type.

[3] Following `mathlib` convention, I do not take closure in the definition of $\operatorname{supp} f$. This way it works without a topological space structure on the domain of f.

- for a locally Lipschitz map, the Hausdorff dimension of the image of a set is less than or equal to the Hausdorff dimension of the original set;
- if s is a set in $\mathbb{R}^n$ such that $\dim_H s < n$, then the complement to s is everywhere dense.

I have formalized the definition of the Hausdorff dimension of a set in a metric space, and I am going to formalize the rest in the next few weeks, then start working on the proof of the weak Whitney embedding theorem.

Once this project is done, it opens path to formalization of large parts of differential geometry. For example, it would be nice to formalize differential forms on a manifold and integral of a differential form over a manifold; the latter part needs a partition of unity subordinate to the atlas. Another possible goal is to formalize Sard's Theorem and related theorems like Thom's transversality theorem.

Acknowledgements

TODO