SPECTRA OF ORDER-3 EQUATIONS

Definition. The *spectrum* of an equation L is the set of nonnegative integers n such that there exists a magma of order n satisfying L.

Using definability to eliminate cases and looking at laws up to order 3, `https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Tarski's.20axiom.20543/near/482210842` indicates that the only equations we must consider are equations 1, 2, 14, 40, 43, 63, 66, 73, 115, 118, 125, 167, 168, 313, 332, 335 (from the list at `https://teorth.github.io/equational_theories/implications/`). Eliminating the trivial equation 2 and all equations that are known to have full ($\mathbb{Z}_{\geq 0}$) spectrum, we obtain the list of equations 63, 66, 73, 115, 118, 125, 167, and 168. However, 73 is equivalent to 125 and to the dual of 118 for finite magmas, so all three have the same spectrum.

Furthermore, it turns out that 63 and 73 have the same spectrum, which follows from the following:

Proposition 1. Let $(S, \diamond)$ be a finite left-cancellative magma. Then $(S, \diamond)$ satisfies 63 if and only if $(S, \diamond')$ satisfies 73, where $x \diamond' y = L_x^{-1} y$.

Since both 63 and 73 imply left-cancellativity, it follows that both have the same spectrum.

Proof. It will be easier instead of working with 73 to work with 229, which is equivalent for finite magmas. 63 states that if $x \diamond y = z$ and $x \diamond z = w$, then $y \diamond w = x$. Expressing in terms of $\diamond'$, if $x \diamond' z = y$ and $x \diamond' w = z$, then $y \diamond' x = w$, and thus for any w, x,

$$(x \diamond' (x \diamond' w)) \diamond' x = w,$$

so $(S, \diamond')$ satisfies 229.

Conversely, suppose $(S, \diamond')$ satisfies 229. Then if $y \diamond' x = z$ and $y \diamond' z = w$, then $w \diamond' y = x$. Thus if $y \diamond z = x$ and $y \diamond w = z$, then $w \diamond x = y$, and thus for any w, y,

$$w \diamond (y \diamond (y \diamond w)) = w,$$

so $(S, \diamond)$ satisfies 63. $\square$

The remaining list of potentially interesting equations is the following:

63. $x = y \diamond (x \diamond (x \diamond y))$
66. $x = y \diamond (x \diamond (y \diamond y))$
115. $x = y \diamond ((x \diamond x) \diamond y)$
167. $x = (y \diamond x) \diamond (x \diamond y)$
168. $x = (y \diamond x) \diamond (x \diamond z)$

It is known that all of these equations have nontrivial non-full spectrum. The first three have magmas of size 3 but not 2, and the last two have magmas of size 4 but not 3 or 2.

168 is the central groupoid law, and it is known that its spectrum is $\{n^2 : n \in \mathbb{Z}_{\geq 0}\}$. Most of the remainder of this writeup will be dedicated to the two equations 66 and 167, for which we can determine the spectrum.

It is possibly noteworthy that for the remaining two equations (63 and 115), studying linear magmas $x \diamond y = ax + by$ over (say) a finite field yields solutions when a certain quintic, factoring into a quadratic and a cubic, has roots. For equations 66 and 167, the corresponding equation factors into two quadratics. The existence of the cubic means it may be more difficult or impossible to find modular obstructions to the equations' spectra, which may make those cases more difficult to tackle. Indeed, we show later that there are no modular obstructions to 115 having solutions.

Proposition 2. Equation 66, $x = y \diamond (x \diamond (y \diamond y))$, has a satisfying magma on n elements if and only if there exists a Mendelsohn triple system of order n; as a result, its spectrum is

$$\{n \in Z_{\geq 0} : n \equiv 0, 1 \pmod 3\} - \{6\}.$$

Note. A Mendelsohn triple system is an ordered generalization of a Steiner triple system. In particular, a Mendelsohn triple system (MTS) on a set S is a set T of ordered triples (x, y, z) with x, y, z distinct elements of S, which satisfies:

(1) T is cyclically closed, in the sense that if $(x, y, z) \in T$, then $(y, z, x), (z, x, y) \in T$
(2) For every pair $x, y \in S$, $x \neq y$, there is exactly one z such that $(x, y, z) \in T$.

In other words, an MTS is a partition of the complete digraph on the elements of S into cyclic triangles.

It is known that an MTS of size n exists if and only if $n \equiv 0$ or $1 \pmod 3$ and $n \neq 6$. [1]

Proof. First suppose an $MTS(n)$ exists; call it T.

Define the magma on $[n]$ by the equations

$$x \diamond y = \begin{cases} x & x = y \\ z & (x, y, z) \in T \end{cases}$$

By the properties of the MTS, exactly one of these holds. Furthermore, it is easy to check that equation 66 is satisfied: if $x = y$ then all intermediate products are x, and if $x \neq y$ then $y \diamond (x \diamond (y \diamond y)) = y \diamond (x \diamond y) = y \diamond z = x$ if (x, y, z) (and thus (y, z, x)) is in T.

Conversely, suppose there exists a magma M of size n satisfying equation 66 on $[n]$. Letting $S : M \to M$, $Sx = x \diamond x$ be the squaring operator, and L_x and R_x being the operations of left- and right-multiplication by x, we see that equation 66 states that

$$L_y R_{Sy} = Id.$$

As M is finite, L_y and R_{Sy} must both be invertible. Substituting $y \diamond x$ for x in 66, $y \diamond x = y \diamond ((y \diamond x) \diamond (y \diamond y))$, and as L_y is invertible we may cancel y from both sides to obtain $x = (y \diamond x) \diamond (y \diamond y)$. Letting $x = y$, $x = (x \diamond x) \diamond (x \diamond x)$, so $S^2 = Id$, which further implies that S must be invertible. As R_{Sy} is invertible and S is invertible, R_y is invertible for all y.

Equation 66 states that if $x \diamond Sy = z$, then $y \diamond z = x$. Substituting Sy for y and using $S^2 = Id$, if $x \diamond y = z$, then $Sy \diamond z = x$. Applying this operation three times we obtain that if $x \diamond y = z$, then

- $Sy \diamond z = x$
- $Sz \diamond x = Sy$
- $Sx \diamond Sy = Sz$

Thus $S(x \diamond y) = Sx \diamond Sy$, and so S is a magma automorphism.

Define a triple system T on $[n]$ as follows: $(x, y, z) \in T$ if $x \neq y$ and $x \diamond y = Sz$. Note that if $x = z$, $x \diamond y = Sz = x \diamond x$ implies $x = y$ as L_x is invertible, and similarly for if $y = z$, so all of x, y, z are distinct.

Note that property 2 of an MTS is clearly satisfied, as for all $x \neq y$, $(x, y, S(x \diamond y)) \in T$ (using $S^2 = Id$).

Furthermore, if $x \diamond y = Sz$, using the logic from earlier with z replaced by Sz, we have that $Sy \diamond Sz = x$. As S is a homomorphism, $S(y \diamond z) = x$, so since $S^2 = id$, $y \diamond z = Sx$. Therefore, if $(x, y, z) \in T$, then $(y, z, x) \in T$ (and thus $(z, x, y) \in T$ as well), so T forms an MTS of order n. This completes the proof. $\square$

Proposition 3. The spectrum of equation 167, $x = (y \diamond x) \diamond (x \diamond y)$, is

$$\{n \in \mathbb{Z}_{\geq 0} : n \equiv 0, 1 \pmod 4\}.$$

Proof. Let M be a magma of order n satisfying 167. Let $f : M \times M \to M \times M$ be defined as $f(x, y) = (x \diamond y, y \diamond x)$. Then applying 167 shows that $f^2(x, y) = (y, x)$, and thus $f^4(x, y) = (x, y)$. Thus the action of f partitions $M \times M$ into orbits of order dividing 4, and if $x \neq y$, then $f^2(x, y) \neq (x, y)$, so the orbit of f on (x, y) must be of order 4.

This implies that the action of f partitions $S := \{(x, y) : x, y \in M, x \neq y\}$ into orbits of size exactly 4. Thus 4 divides $|S| = n(n-1)$, so $n \equiv 0, 1 \pmod 4$.

Conversely, suppose $n \equiv 1 \pmod 4$, and let M be a set of size n. Then $2 | \binom{n}{2}$. For $i = 1, \ldots, \frac{n(n-1)}{4}$, define $a_i, b_i, c_i, d_i \in M$ such that every subset $\{x, y\} \subset M$ of size 2 is equal to exactly one of either an $\{a_i, b_i\}$ or a $\{c_i, d_i\}$ (for example, write the $\binom{n}{2}$ pairs in any order, partition that list in half, and make one the (a_i, b_i) and the other the (c_i, d_i)). Then define

- $x \diamond x = x$
- $a_i \diamond b_i = c_i$
- $b_i \diamond a_i = d_i$
- $c_i \diamond d_i = b_i$
- $d_i \diamond c_i = a_i$.

By the definition of a_i, b_i, c_i, d_i, this defines every possible product exactly once, and it is easy to see that 167 holds. □

Finally, we show that there are no modular obstructions to equation 115.

Proposition 4. Every arithmetic progression of nonnegative integers contains an element of the spectrum of 115.

Proof. Suppose for the sake of contradiction that $\{ax + b, a, b \in \mathbb{Z}^+\}$ contains no element of the spectrum of 115. Note that Mendelsohn triple systems give 115-magmas in the same way that they give 66-magmas, as our constructed magmas are idempotent. Thus the spectrum of 115 contains $\{n \in Z_{\geq 0} : n \equiv 0, 1 \pmod 3)\} - \{6\}$, and it follows that $3|a$ and $b \equiv 2 \pmod 3$.

Now, considering linear magmas $x \diamond y = ax + by$ over a commutative ring, we may derive that the resulting magma satisfies 115 if and only if $a = -b^2$ and $b^5 - b^4 - 1 = 0$. The latter quintic factors as $(b^2 - b + 1)(b^3 - b - 1)$, so if p is a prime such that $b^3 - b - 1$ has a solution mod p, then p is in the spectrum of 115.

Take p be a sufficiently large prime with $p \equiv 68 \pmod{69}$. Then $\left(\frac{69}{p}\right) = 1$ and $p \equiv 2 \pmod 3$ (meaning all cube roots exist), so the root

$$\frac{\sqrt[3]{9 + \sqrt{69}} + \sqrt[3]{9 - \sqrt{69}}}{\sqrt[3]{18}}$$

of $b^3 - b - 1$ exists modulo p. Thus p is in the spectrum of 115.

Let $k \equiv bp^{-1} \pmod a$. Then $k \equiv 1 \pmod 3$ and thus k is in the spectrum of 115. Thus the product $kp \equiv b \pmod a$ is in the spectrum of 115, a contradiction. □

References

[1] N. Mendelsohn, *A Natural Generalization of Steiner Triple Systems*, Computers in Number Theory, (1971)