The order-6 law 86082 $x = y \diamond (z \diamond ((y \diamond z) \diamond (w \diamond (x \diamond w))))$ characterizes the operation $x \diamond y = x + iy$ in $\mathbb{Z}[i]$-modules (with $i^2 = -1$). Other order-6 laws also work.

Let $i_x(y) = L_x \circ R_x(y) = x \diamond (y \diamond x)$. The law can be written as $L_y \circ L_z \circ L_{y\diamond z} \circ i_w = \mathrm{id}$. We first prove that $i_v = i_w$ for all v, w. We calculate

$$(x \diamond y) \diamond i_z(u) \overset{86082}{=} L_x \circ L_y \circ L_{x\diamond y} \circ i_{i_z(u)}\big((x \diamond y) \diamond i_z(u)\big)$$
$$= L_x \circ L_y \Big(L_{x\diamond y} \circ L_{i_z(u)} \circ L_{(x\diamond y)\diamond i_z(u)} \circ i_z(u)\Big) \overset{86082}{=} x \diamond (y \diamond u) \tag{1}$$

Using this identity $L_{x\diamond y} \circ i_z = L_x \circ L_y$ in the equation yields $L_y \circ L_z \circ L_y \circ L_z = \mathrm{id}$. We learn in particular that left-multiplication is bijective.

Taking $u = x \diamond y$ in (1) and further left-multiplying by z yields

$$z = (L_z \circ L_{x\diamond y})^2(z) = z \diamond (x \diamond (y \diamond (x \diamond y))). \tag{2}$$

Invertibility of L_z means that

$$x \diamond (y \diamond (x \diamond y)) = x \diamond i_y(x) \text{ is independent of } x \text{ and } y. \tag{3}$$

Invertibility of L_x allows us to deduce $i_y(x)$ is y-independent (but depends on x). We henceforth drop the subscript on the function i.

Next, we define the operation $x - y = x \diamond i(y)$. We shall show that the magma equipped with $-$ is an abelian group with subtraction and that i is a group morphism squaring to negation.

The observation (3) states that $x - x = x \diamond i(x) = x \diamond (x \diamond (x \diamond x))$ is independent of x. We denote $0 = x - x$. Observe that (2) becomes $z = z \diamond 0$ so that zero is a right-unit. In particular,

$$i(x) = i_0(x) = 0 \diamond (x \diamond 0) = 0 \diamond x \tag{4}$$

and $i(0) = 0$. Observe that $i(i(x)) = 0 \diamond i(x) = 0 - x$. We call this operation negation and denote it simply by $-x$.

Taking $z = 0$ in the identity $L_y \circ L_z \circ L_y \circ L_z = \mathrm{id}$ yields $y - (y - x) = x$. In particular for $y = 0$ we see that $-(-x) = x$, namely $i(i(i(i(x)))) = x$.

It is then natural to introduce addition as

$$x + y = x - (-y) = x \diamond i(i(i(y))) = x \diamond (0 \diamond (0 \diamond (0 \diamond y))). \tag{5}$$

One has

$$x + 0 = x, \qquad 0 + x = 0 \diamond i(i(i(x))) = i(i(i(i(x)))) = x. \tag{6}$$

In terms of addition, $x - y = x - (-(-y)) = x + (-y)$, and the magma operation is expressed as $x \diamond y = x \diamond i(i(i(i(y)))) = x + i(y)$.

The technical identity (1) evaluated at $x = 0$ tells us that i is a magma morphism,

$$i(y) \diamond i(u) = i(y \diamond u), \tag{7}$$

and thus $i(x+y) = i(x) + i(y)$. Its square, negation, also is, so that $-(x+y) = (-x) + (-y)$.

We now return to the main equation with (y, z) replaced by $(-y, i(z))$, write it in terms of addition, and use that i is a morphism, to get $x = -y + (-z + ((y+z)+x))$. Using that $t-(t-u) = u$, and how subtraction and addition are related through the negation involution, we get

$$z + (y + x) = (y + z) + x. \tag{8}$$

For $x = 0$ this yields commutativity $z+y = y+z$. Then, by swapping operands around, we get associativity $(x + y) + z = x + (y + z)$. Together with what we know about negation and 0, we deduce that $(M, +)$ is an abelian group. It is equipped with a group automorphism i with $i(i(x)) = -x$, which completes the description of M as a $\mathbb{Z}[i]$-module.