ON UNIQUENESS OF σ-FINITE MEASURES ON A PRODUCT SPACE

by Etienne Marion

Consider $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ two σ-finite measure spaces. It is well known that the product measure $\mu \otimes \nu$ is the only measure ξ over the product space $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ satisfying

$$\forall E \in \mathcal{A}, F \in \mathcal{B}, \xi(E \times F) = \mu(E)\nu(F).$$

Thus the product of two σ-finite measures is characterized by its value on what we will call *measurable rectangles*. This however is no longer true if μ and ν are only assumed to be s-finite, i.e. to be a countable sum of finite measures. Indeed, consider $\mu = \nu := \infty \cdot \mathrm{Leb}_{[0,1]}$, i.e. the measure which to a Lebesgue-measurable set associates ∞ if it has positive Lebesgue measure, and zero otherwise. Denote by λ the Lebesgue measure on the diagonal of $[0,1]^2$. Then for any $E, F \in \mathcal{L}([0,1])$ (the Lebesgue σ-algebra), if either E or F has Lebesgue measure 0, then

$$(\mu \otimes \nu)(E \times F) = (\mu \otimes \nu + \lambda)(E \times F) = 0.$$

Otherwise, both measures give infinite measure to $E \times F$, therefore the two measures $\mu \otimes \nu$ and $\mu \otimes \nu + \lambda$ coincide on measurable rectangles. However $\mu \otimes \nu$ gives measure zero to the diagonal, while $\mu \otimes \nu + \lambda$ gives it measure $\sqrt{2}$, so the two measures are different.

Consider now the more general case of a σ-finite measure μ defined over the product space $(X \times Y, \mathcal{A} \otimes \mathcal{B})$. Is it characterized by its value on measurable rectangles? This is true if μ is assumed to be finite, as can be shown via the π-λ theorem. It is also true if μ is the product of two σ-finite measures, as discussed above. However, it turns out that it is not true in general. The goal of this note is to provide a counter example. This counter example was presented to me by Sébastien Gouëzel, and is based on problem 14E in *Problems for Mathematicians, Young and Old* by Paul R. Halmos.

We will build a function $B : \mathbb{R}^2 \to \mathbb{R}_+$ such that for any $E, F \in \mathcal{L}(\mathbb{R})$ with positive measure, B has an infinite integral over $E \times F$ against the Lebesgue measure on $\mathbb{R}^2$. Therefore, using the same trick as before, it will not be possible to distinguish the measure with density B from the same measure to which we add the Lebesgue measure over the diagonal of $\mathbb{R}^2$ by simply looking at their values over measurable rectangles. On the other hand, because B is finite everywhere, the measure with density B will be σ-finite (take $X_n := \{x \mid B(x) \leq n\} \cap [-n, n]^2$ to get a sequence of spanning sets with finite measure). In what follows, if $E \in \mathcal{L}(\mathbb{R})$, we will denote by $|E|$ the Lebesgue measure of E.

To construct B, consider $(r_n)_{n \in \mathbb{N}}$ a dense sequence in $\mathbb{R}$, and let I_n be the interval centered at r_n and with length $\frac{1}{2^n}$. We set

$$\begin{aligned} A : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sum_{n \geq 0} 4^n \mathbb{1}_{I_n}(x). \end{aligned}$$

Because $\sum_{n \geq 0} |I_n| < \infty$, the Borel-Cantelli lemma implies that for Leb-almost every x, the sum defining $A(x)$ contains only a finite number of non-zero terms, and is therefore finite. We redefine A by setting it to 0 on the zero-measure set where it is infinite.

We now set $B(x, y) \coloneqq A(x - y)$. Let $E, F \in \mathcal{L}(\mathbb{R})$ with positive measure. Doing the change of variable $u = x - y$ and $v = y$, we have

$$\begin{aligned}\int_{E\times F} B(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^2} A(x-y)\mathbb{1}_{E\times F}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} A(u)\mathbb{1}_{E\times F}(u+v,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{\mathbb{R}} A(u)\int_{\mathbb{R}} \mathbb{1}_{E\times F}(u+v,v)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_{\mathbb{R}} A(u)\int_{\mathbb{R}} \mathbb{1}_{(E-u)\cap F}(v)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_{\mathbb{R}} A(u)|(E-u)\cap F|\,\mathrm{d}u.\end{aligned}$$

We will now prove that, for some constant $c > 0$, there exist arbitrarily large n such that for every $u \in I_n$, $|(E - u) \cap F| \geq c$. This will imply that

$$\int_{E\times F} B(x,y)\,\mathrm{d}x\,\mathrm{d}y \geq c\int_{I_n} A(x)\,\mathrm{d}x \geq c4^n|I_n| = c2^n,$$

which in turn implies that $\int_{E\times F} B(x,y)\,\mathrm{d}x\,\mathrm{d}y = \infty$. To do that, recall that because E and F both have positive measure, the Lebesgue's density theorem implies that there exist $x \in E$, $y \in F$ and $\varepsilon > 0$ such that

$$|E \cap [x-\varepsilon, x+\varepsilon]| > \frac{4\varepsilon}{3} \text{ and } |F \cap [y-\varepsilon, y+\varepsilon]| > \frac{4\varepsilon}{3}.$$

Let $u \in \left[x - y - \frac{\varepsilon}{7}, x - y + \frac{\varepsilon}{7}\right]$. Then

$$|(E-u)\cap[x-u-\varepsilon, x-u+\varepsilon]| = |E\cap[x-\varepsilon,x+\varepsilon]| \geq \frac{4\varepsilon}{3}.$$

Moreover, $x - u \in \left[y - \frac{\varepsilon}{7}, y + \frac{\varepsilon}{7}\right]$, which means that

$$\begin{aligned}|(E-u)\cap[y-\varepsilon,y+\varepsilon]| &\geq |(E-u)\cap[x-u-\varepsilon,x-u+\varepsilon]\cap[y-\varepsilon,y+\varepsilon]| \\ &= |((E-u)\cap[x-u-\varepsilon,x-u+\varepsilon]) \setminus \\ &\qquad ([y-\varepsilon,y+\varepsilon]^c\cap[x-u-\varepsilon,x-u+\varepsilon])| \\ &\geq |(E-u)\cap[x-u-\varepsilon,x-u+\varepsilon]| - \\ &\qquad |[y-\varepsilon,y+\varepsilon]^c\cap[x-u-\varepsilon,x-u+\varepsilon]| \\ &\geq \frac{4\varepsilon}{3} - \frac{2\varepsilon}{7} > \varepsilon.\end{aligned}$$

We also know that $|F \cap [y - \varepsilon, y + \varepsilon]| > \frac{4\varepsilon}{3}$. Because $|[y - \varepsilon, y + \varepsilon]| = 2\varepsilon$, we deduce that

$$|(E-u)\cap F| \geq |(E-u)\cap F\cap[y-\varepsilon,y+\varepsilon]| > \frac{7\varepsilon}{3} - 2\varepsilon = \frac{\varepsilon}{3}.$$

Take $c \coloneqq \frac{\varepsilon}{3}$. To conclude, we notice that because the sequence $(r_n)_{n\in\mathbb{N}}$ is dense, the interval $\left[x - y - \frac{\varepsilon}{8}, x - y + \frac{\varepsilon}{8}\right]$ contains infinitely many terms of the sequence, which implies that there exist arbitrarily large n such that $\left[x - y - \frac{\varepsilon}{7}, x - y + \frac{\varepsilon}{7}\right]$ contains I_n. ■