Lean-SMT: Integration of Theorem Provers with SMT Solvers

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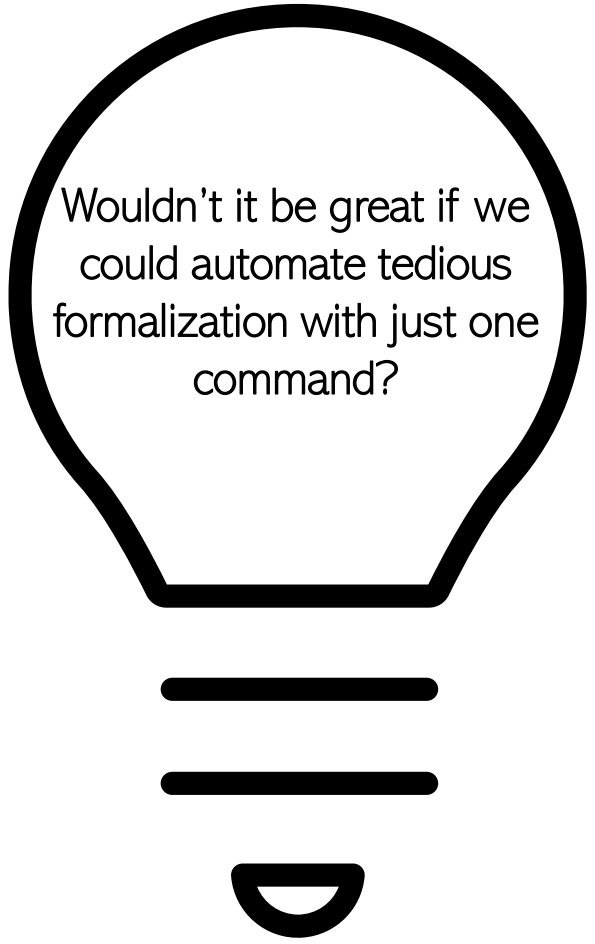
Motivation

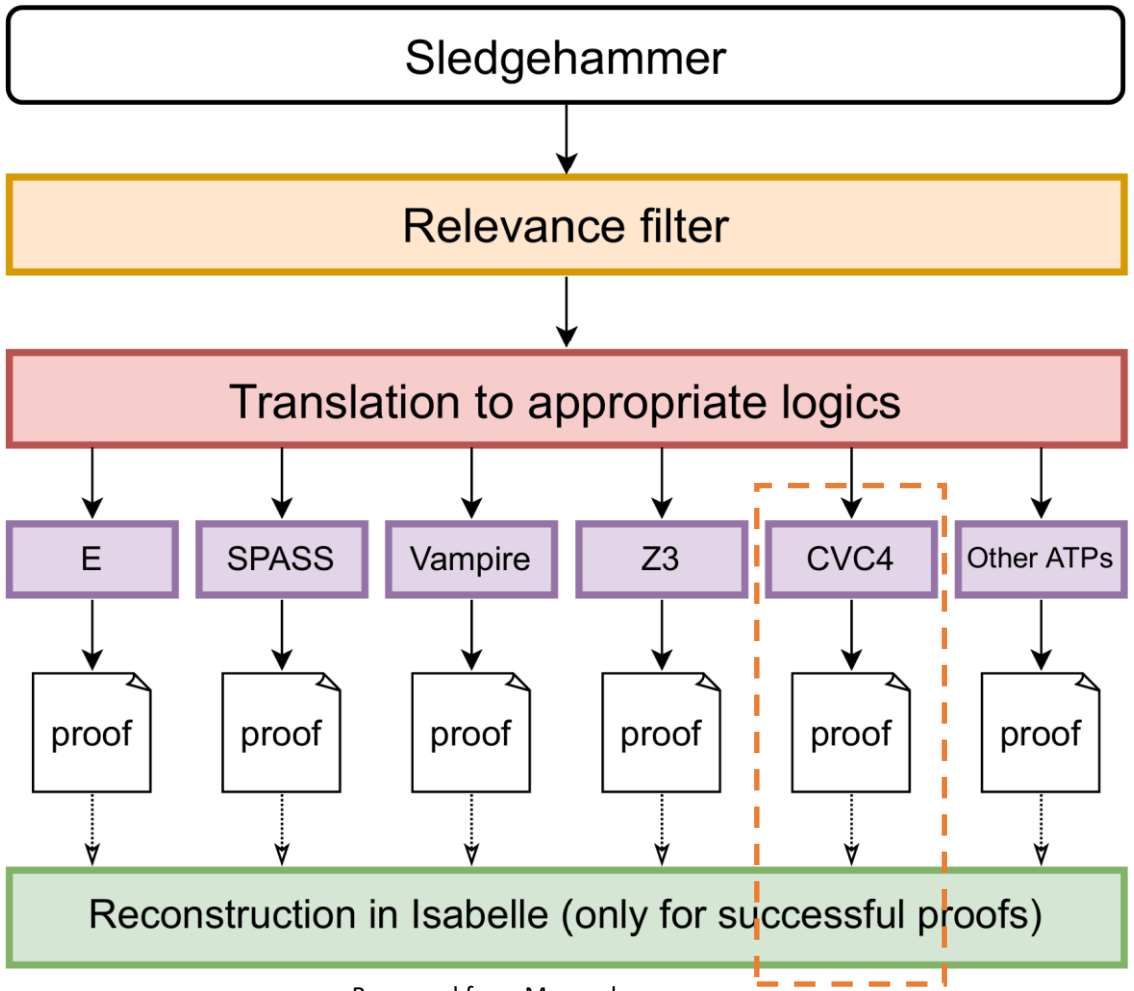
Suppose we encounter this in the Lean theorem prover:

```
x z: ℤ
hxz: x + z < 2
f: ℤ → ℤ
hz: 0 < z
hx: 0 ≤ x
⊢ ∀ (y : ℤ), f (x + y) = f y
```



Hammers

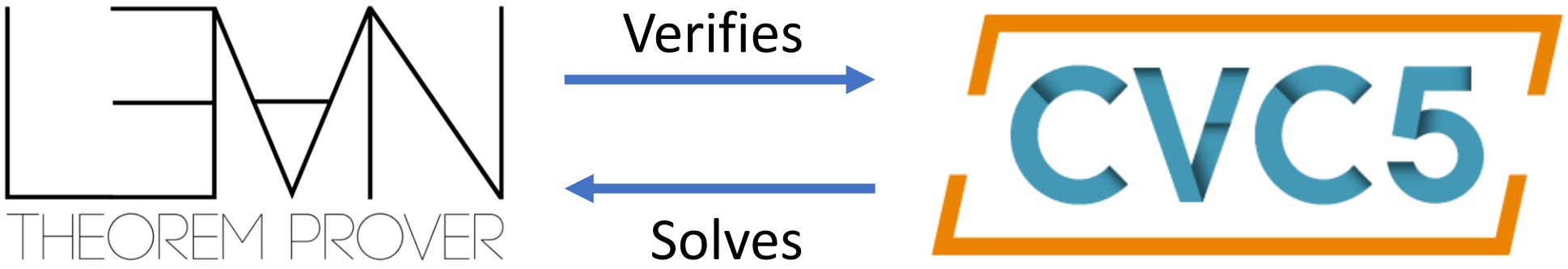
- Powerful automation tactics
  - 40% of Mizar Math library
  - 47% of Flyspeck libraries (HOL Light)
  - Similar results in Isabelle
- Integrate ATPs into proof assistants
  - SMT, Superposition, tableau
- Components
  - Premise selector
  - Translation module (into ATP's logic)
  - Proof reconstruction module
- Implement highlighted portion in Lean
- Use cvc5!



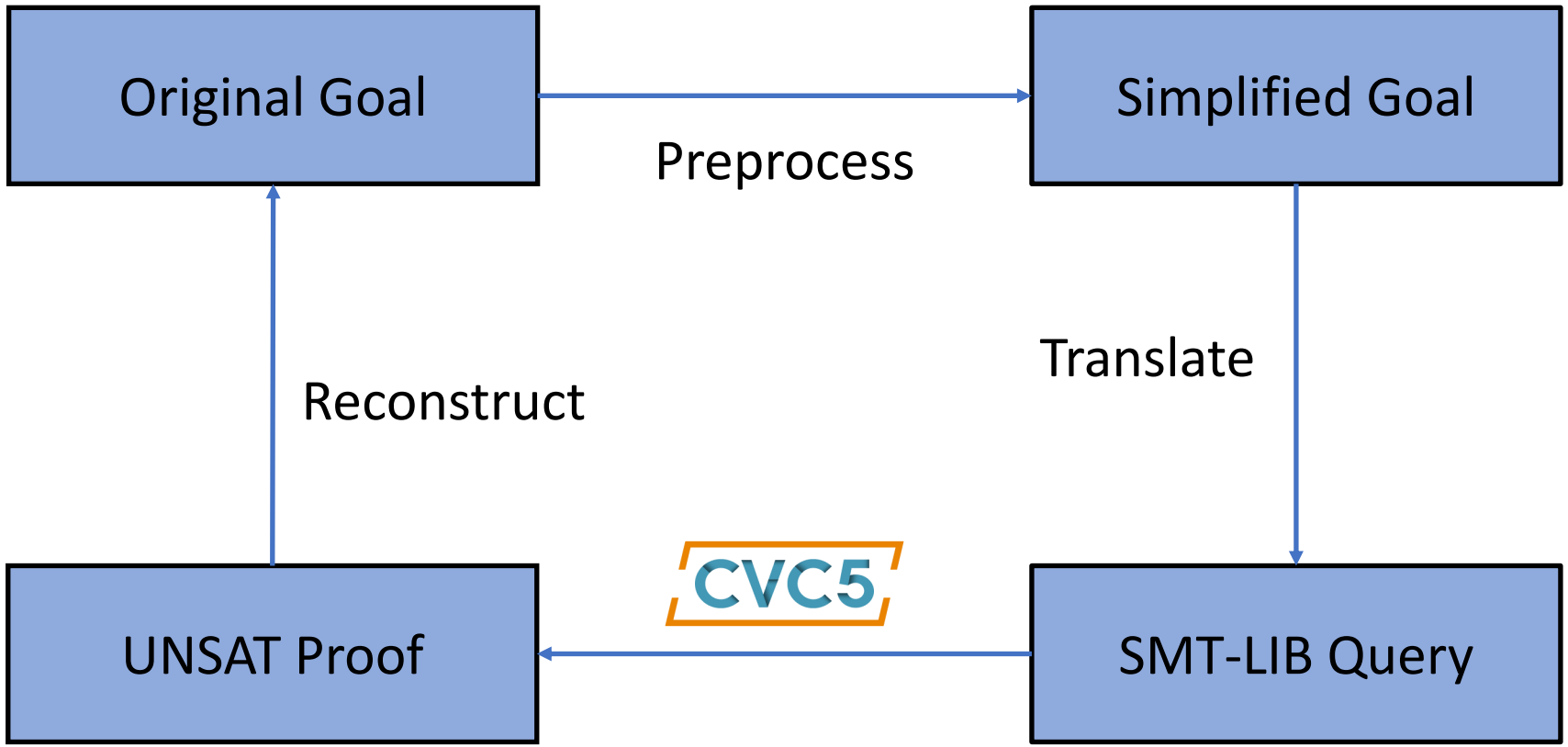
Borrowed from Magnushammer paper.

The Lean-SMT Tactic

A tactic that integrates cvc5 and Lean would benefit both worlds



Tactic Workflow



Lean-SMT's Pipeline

Preprocessing → Translation → Solving → Reconstruction

Original Lean Goal

```
⊢ ∀ (G : Type u) [Group G] (e : G),
    (∀ (a : G), e * a = a) ↔ e = 1
```

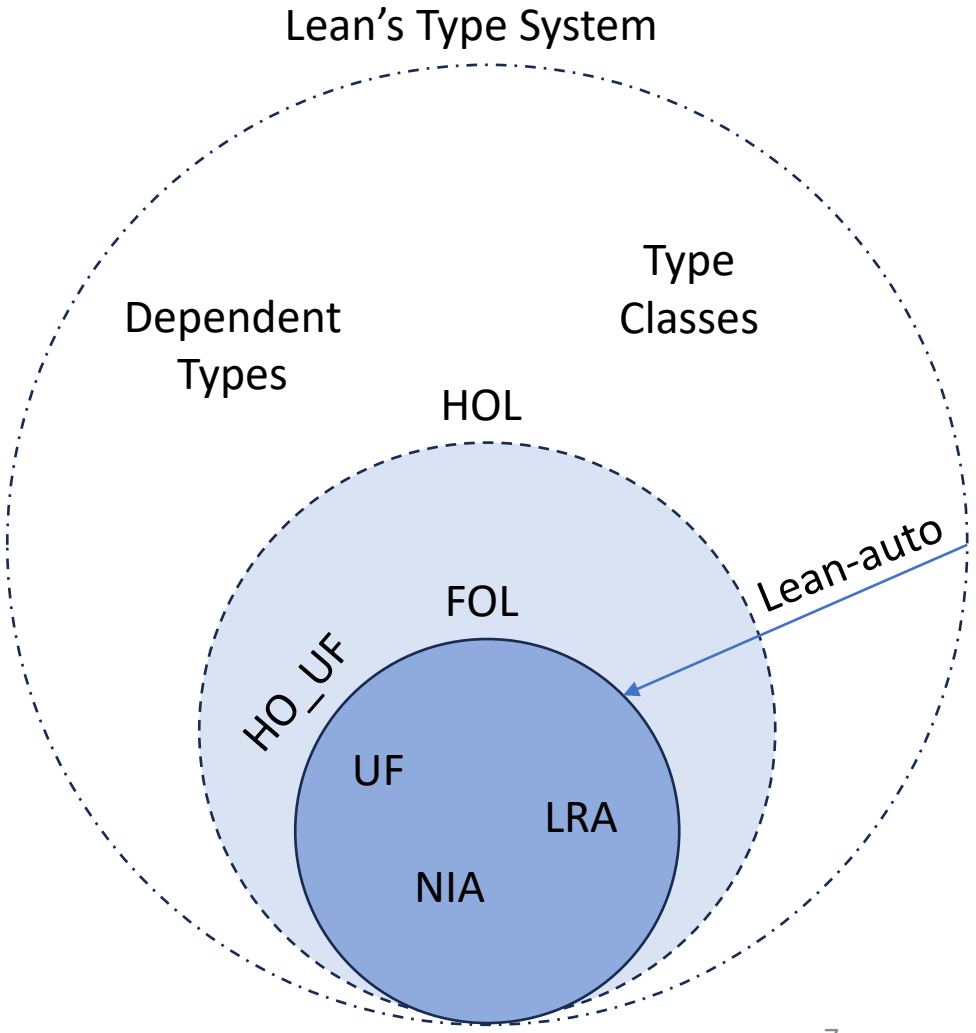
SMT-LIB does not support quantifying over Types and Type classes!!!

Identity element is unique in any group G!

Lean-SMT's Pipeline

Preprocessing → Translation → Solving → Reconstruction

- Lean's type system is very rich!
- SMT-LIB is based on FOL (with limited HOL)
- Utilize Lean-auto!
  - Very helpful, but not enough!
- Theories are minimal
  - Do not support every operation and sort
- Reduce papercuts:
  - Replace $p \leftrightarrow q$ with $p = q$
  - Embed `Bool` into `Prop`
  - Embed `Nat` into `Int`
  - Etc.



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Preprocessing | Translation | Solving | Reconstruction

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Preprocessed Lean Goal

```
G: Type u
inst: Group G
e e': G
op: G → G → G
inv: G → G
one_mul: ∀ (a : G), op e a = a
inv_mul_cancel: ∀ (a : G), op (inv a) a = e
mul_assoc: ∀ (a b c : G),
 op (op a b) c = op a (op b c)

⊢ (∀ (a : G), (op e' a = a)) = (e' = e)
```

Lean-SMT's Pipeline

Preprocessing → **Translation** → Solving → Reconstruction

Preprocessed Lean Goal

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 op (op a b) c = op a (op b c)

⊢ (∀ (a : G), (op e' a = a)) = (e' = e)
```

SMT-LIB Query

```
(declare-sort G 0)
(declare-const e G)
(declare-const |e'| G)
(declare-fun op (G G) G)
(declare-fun inv (G) G)
(assert (forall ((a G)) (= (op e a) a)))
(assert (forall ((a G))
  (= (op (inv a) a) e)))
(assert (forall ((a G) (b G) (c G))
  (= (op (op a b) c)
     (op a (op b c))))))
(assert (distinct
  (forall ((a G)) (= (op |e'| a) a)
  (= |e'| e))))
(check-sat)
```

Translation is sound because G is nonempty!

Lean-SMT's Pipeline

Preprocessing → Translation → **Solving** → Reconstruction

SMT-LIB Query

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  (= |e'| e))))
(check-sat)
```

cvc5 Proof (CPC)

```
(define @t1 () (eo::var "a" G))
(define @t2 () (op e @t1))
...
(assume @p1 (forall @t3 (= @t2 @t1)))
(assume @p2 (forall @t3 (= @t4 e)))
(assume @p3 @t8)
(assume @p4 (not (= (forall @t3 (= @t9 @t1))
                    (= |e'| e))))
(step @p5 :rule quant-merge-prenex
          :args ((= @t8 @t10)))
(step @p6 :rule eq_resolve
          :premises (@p3 @p5))
(step @p7 :rule eq-symm :args (@t4 e))
(step @p8 :rule cong :premises (@p7)
          :args ((forall ((a G)))))
...
```

Lean-SMT's Pipeline

Preprocessing → Translation → Solving → **Reconstruction**

cvc5 Proof (CPC)

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(define @t1 () (eo::var "a" G))
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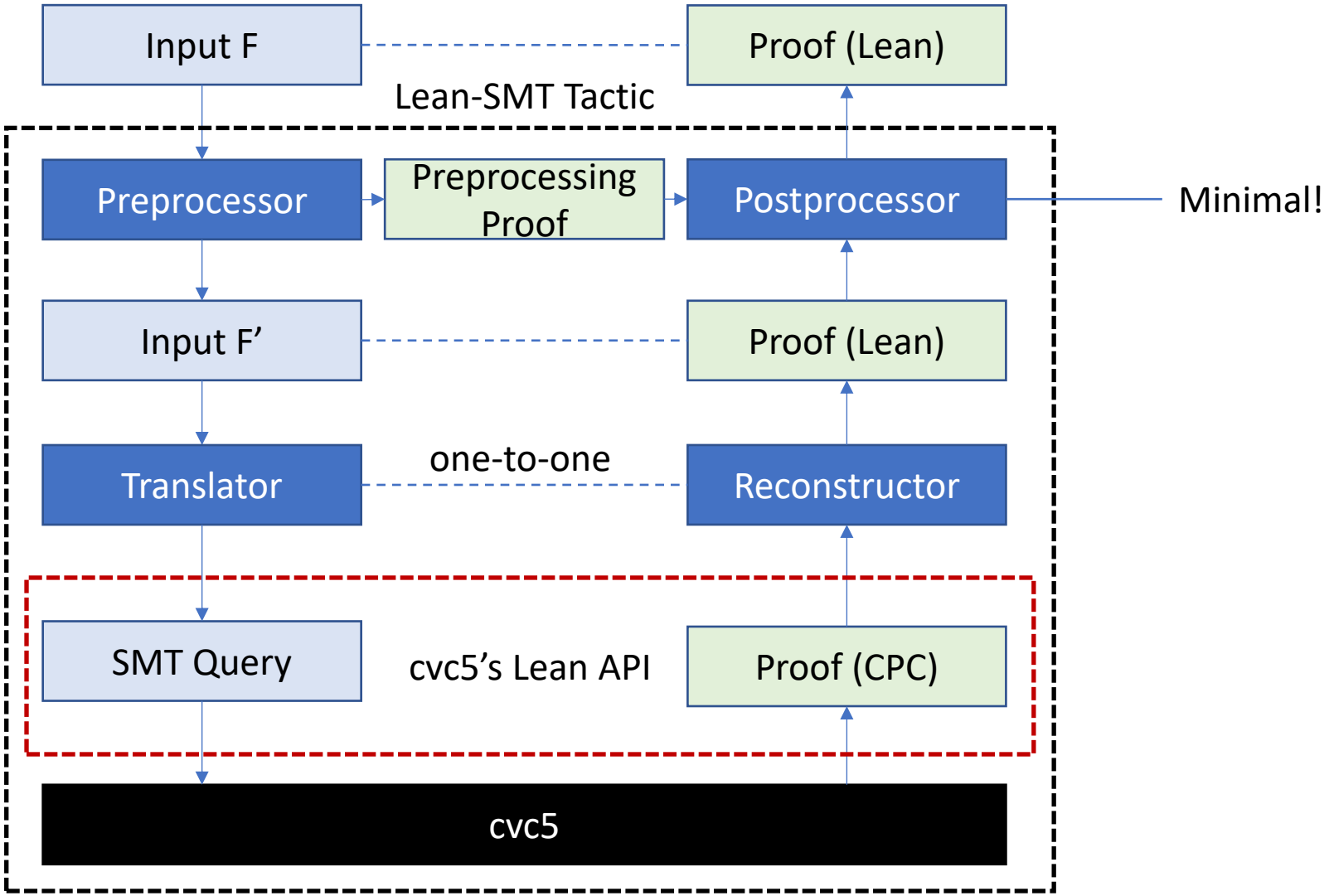
...
```

Lean Proof

```
have a0 : e = op (inv e') e' := by assumption
have a1 : e' = op e' e' := by assumption
...
have s2 : inv e' = inv e' := Eq.refl (inv e')
have s3 : op e' e' = e' := Eq.symm a1
have s4 : op (inv e') (op e' e') =
          op (inv e') e' :=
  congr (congr (Eq.refl op) (Eq.refl (inv e')))
        (Eq.symm a.17)
have s5 : op (inv e') e' = e := Eq.symm a0
have s6 : op (inv e') (op e' e') = e :=
  Eq.trans (congr (congr (Eq.refl op)
                         (Eq.refl (inv e')))
                  (Eq.symm a1)) (Eq.symm a0)
have s7 : e' = e' := Eq.refl e'
...
```

Check with Lean's Kernel!!!

Overall Architecture



Reconstruction: Proof Replay vs. Verified Checker

Proof Replay

- Reconstructs proofs inference-by-inference inside the proof assistant
- Example: Isabelle SledgeHammer
- Pro: easy to implement and modify
- Con: slower
- Our approach!
  - No standard SMT proof format
  - cvc5 proof rules are not stable

Verified Checker

- Proofs are checked within the proof assistant yielding a proof by reflection rather than simulation
- Example: SMTCoq
- Pro: fast
- Con: hard to implement and modify

Proof Reconstruction

- A cvc5 proof is a sequence of steps (instantiations of proofs rules)
- There are ~130 proof rules and ~360 rewrite rules
- Proof and rewrite rules must be formalized in Lean!
- Three approaches:
  - Theorems: prove a theorem capturing a proof rule/rewrite rule
  - Tactics: repeated application of a set of theorems
  - Reflection: normalization steps

Proof Reconstruction: Theorems

The most straightforward approach is to formalize and prove a theorem capturing a proof rule/rewrite rule (e.g. **RESOLUTION**)

$$\frac{C_1, C_2 \mid pol, L}{C}$$

where

- C_1 and C_2 are CNF clauses
- pol is either $\top$ or $\bot$, representing the polarity of the pivot on the first clause
- L is the pivot of the resolution, which occurs as is (resp. under a NOT) in C_1 and negatively (as is) in if $pol = \top$ ($pol = \bot$)

```
theorem orN_resolution
  (hps : orN ps) (hqs : orN qs)
  (hi : i < ps.length)
  (hj : j < qs.length)
  (hij : ps[i] = ¬qs[j]) :
  orN (ps.eraseIdx i ++ qs.eraseIdx j)
```

Proof Reconstruction: Tactics

- In some cases where the formalization of the process is too complex, we write a tactic
- Example: **QUANT_MINISCOPE_AND** rewrite rule

$$\forall X. F_1 \wedge \cdots \wedge F_n = (\forall X. F_1) \wedge \cdots \wedge (\forall X. F_n)$$

- X denotes a multiple variables!
- This involves two key theorems

```
theorem miniscope_andN {ps : List (α → Prop)} :
  (∀ x, andN (ps.map (· x))) = andN (ps.map (∀ x, · x))
theorem forall_congr {α : Sort u} {p q : α → Prop} (h : ∀ a, p a = q a) :
  (∀ a, p a) = (∀ a, q a)
```

- Iteratively apply the two theorems, peeling one variable at a time

```
example : (∀ x y, p x y ∧ q x y) = ((∀ x y, p x y) ∧ (∀ x y, q x y)) := by
  smt
```

Proof Reconstruction: Reflection

- Verified programs allow us to prove correctness properties about an executable program
- `poly_norm` normalizes polynomials (up to distrib., assoc., and comm.)

```
example (x y : Int) (z : Real) : 1 * ↑(x + y) * z / 4 = 1 / (2 * 2) * (z * ↑y + ↑x * z) := by
  poly_norm
```

- `poly_norm` completely expands polynomials into a sum of monomials
- Variables in monomials are defined as `List Nat` allowing them to be compared easily
- Our implementation proves theorems pushing evaluation into each operator.

```
theorem denote_mul {m₁ m₂ : Monomial} : (m₁.mul m₂).denote ctx = m₁.denote ctx * m₂.denote ctx
theorem denote_add {p₁ p₂ : Polynomial} : (p₁.add p₂).denote ctx = p₁.denote ctx + p₂.denote ctx
```

- And a correctness theorem

```
theorem denote_eq_from_toPolynomial_eq {e₁ e₂ : RealExpr}
  (h : e₁.toPolynomial = e₂.toPolynomial) : e₁.denote ictx rctx = e₂.denote ictx rctx
```

Demo!

Current Status & Evaluation

- Supports UFs, linear integer/real arithmetic, and quantifiers
- No benchmarks for Lean (in progress!)
- Tested on Isabelle benchmarks from previous publications
- In SMT-LIB format (1 minute timeout)

Benchmark Set	Reconstruction Succeeded	Solved by cvc5	Total Number of Benchmarks
SledgeHammer	1722	1750	**4260**
17 Provers	2762	2812	**5000**
Total	**4484**	**4562**	**9260**

Next Steps

- Create a benchmark set for Lean!
- Support more theories (BitVec, Datatypes, etc.)
- Support more proof rules (current state: 180 rules)
- Add more preprocessing steps (reduce papercuts)
- Improve integration with Lean-auto
- Work towards a Sledgehammer in Lean
  - described as “the difference between walking and running” (L Paulson).

Questions?