## Appendix A. Bounding $\sum_{n\leq a} n^{-s} - \int_0^a \frac{du}{u^s}$, or approximating $\zeta(s)$

We want a good explicit estimate on

$$\sum_{n\leq a} \frac{1}{n^s} - \int_0^a \frac{du}{u^s},$$

for $a$ a half-integer. As it turns out, this is the same problem as that of approximating $\zeta(s)$ by a sum $\sum_{n\leq a} n^{-s}$. This is one of the two[1] main, standard ways of approximating $\zeta(s)$.

The non-explicit version of the result was first proved in [HL21, Lemmas 1 and 2]. The proof there uses first-order Euler-Maclaurin combined with a decomposition of $\lfloor x\rfloor - x + 1/2$ that turns out to be equivalent to Poisson summation. The exposition in [Tit86, §4.7–4.11] uses first-order Euler-Maclaurin and van de Corput's Process B; the main idea of the latter is Poisson summation.

There are already several explicit versions of the result in the literature. In [Che99], [Kad13] and [Sim20], what we have is successively sharper explicit versions of Hardy and Littlewood's original proof. The proof in [DHZA22, Lemma 2.10] proceeds simply by a careful estimation of the terms in high-order Euler-Maclaurin; it does not use Poisson summation. Finally, [dR24] is an explicit version of [Tit86, §4.7–4.11]; it gives a weaker bound than [Sim20] or [DHZA22]. The strongest of these results is [Sim20].

We will give another version here, in part because we wish to relax conditions – we will work with $|\Im s| < 2\pi a$ rather than $|\Im s| \leq a$ – and in part to show that one can prove an asymptotically optimal result easily and concisely. We will use first-order Euler-Maclaurin and Poisson summation. We assume that $a$ is a half-integer; if one inserts the same assumption into [DHZA22, Lemma 2.10], one can improve the result there, yielding an error term closer to the one here.

*Notation.* We recall that $e(\alpha) = e^{2\pi i\alpha}$, and $O^*(R)$ means a quantity of absolute value at most $R$. We write $\sum_n$ for $\sum_{n\geq 1}$, $\sum_{n\leq a}$ for $\sum_{1\leq n\leq a}$, etc.

### A.1. **The decay of a Fourier transform.**

Our first objective will be to estimate the Fourier transform of $t^{-s}\mathbb{1}_{[a,b]}$. In particular, we will show that, if $a$ and $b$ are half-integers, the Fourier cosine transform has quadratic decay *when evaluated at integers*. In general, for real arguments, the Fourier transform of a discontinuous function such as $t^{-s}\mathbb{1}_{[a,b]}$ does not have quadratic decay.

**Lemma A.1.** *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R}\setminus\{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then*

$$\int_a^b t^{-s} e(\nu t)dt = \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i\varphi_\nu'(t)}\right|_a^b + \sigma\int_a^b \frac{t^{-\sigma-1}}{2\pi i\varphi_\nu'(t)} e(\varphi_\nu(t))dt + \int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i(\varphi_\nu'(t))^2} e(\varphi_\nu(t))dt, \quad \text{(A.1)}$$

*where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$.*

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[1] The other one is the approximate functional equation.

*Proof.* We write $t^{-s}e(\nu t) = t^{-\sigma}e(\varphi_\nu(t))$ and integrate by parts with $u = t^{-\sigma}/(2\pi i\varphi_\nu'(t))$, $v = e(\varphi_\nu(t))$. Here $\varphi_\nu'(t) = \nu - \tau/(2\pi t) \neq 0$ for $t \in [a,b]$ because $t \geq a > |\tau|/(2\pi|\nu|)$ implies $|\nu| > |\tau|/(2\pi t)$. Clearly

$$udv = \frac{t^{-\sigma}}{2\pi i\varphi_\nu'(t)} \cdot 2\pi i\varphi_\nu'(t)e(\varphi_\nu(t))dt = t^{-\sigma}e(\varphi_\nu(t))dt,$$

while

$$du = \left(\frac{-\sigma t^{-\sigma-1}}{2\pi i\varphi_\nu'(t)} - \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i(\varphi_\nu'(t))^2}\right)dt.$$

□

**Lemma A.2.** *Let $g : [a,b] \to \mathbb{R}$ be continuous, with $|g(t)|$ non-increasing. Then $g$ is monotone, and $\|g\|_{\mathrm{TV}} = |g(a)| - |g(b)|$.*

*Proof.* Suppose $g$ changed sign: $g(a') > 0 > g(b')$ or $g(a') < 0 < g(b')$ for some $a \leq a' < b' \leq b$. By IVT, there would be an $r \in [a',b']$ such that $g(r) = 0$. Since $|g|$ is non-increasing, $g(b') = 0$; contradiction. So, $g$ does not change sign: either $g \leq 0$ or $g \geq 0$.

Thus, there is an $\varepsilon \in \{-1,1\}$ such that $g(t) = \varepsilon|g(t)|$ for all $t \in [a,b]$. Hence, $g$ is monotone. Then $\|g\|_{\mathrm{TV}} = |g(a) - g(b)|$. Since $|g(a)| \geq |g(b)|$ and $g(a)$, $g(b)$ are either both non-positive or non-negative, $|g(a) - g(b)| = |g(a)| - |g(b)|$. □

**Lemma A.3.** *Let $\varphi : [a,b] \to \mathbb{R}$ be $C^1$ with $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Let $h : [a,b] \to \mathbb{R}$ be such that $g(t) = h(t)/\varphi'(t)$ is continuous and $|g(t)|$ is non-increasing. Then*

$$\left|\int_a^b h(t)e(\varphi(t))dt\right| \leq \frac{|g(a)|}{\pi}.$$

This is a statement of known type ("non-stationary phase").

*Proof.* Since $\varphi$ is $C^1$, $e(\varphi(t))$ is $C^1$, and $h(t)e(\varphi(t)) = \frac{h(t)}{2\pi i\varphi'(t)}\frac{d}{dt}e(\varphi(t))$ everywhere. By Lemma A.2, $g$ is of bounded variation. Hence, we can integrate by parts:

$$\int_a^b h(t)e(\varphi(t))dt = \left.\frac{h(t)e(\varphi(t))}{2\pi i\varphi'(t)}\right|_a^b - \int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i\varphi'(t)}\right).$$

The first term on the right has absolute value $\leq \frac{|g(a)|+|g(b)|}{2\pi}$. Again by Lemma A.2,

$$\left|\int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i\varphi'(t)}\right)\right| \leq \frac{1}{2\pi}\|g\|_{\mathrm{TV}} = \frac{|g(a)| - |g(b)|}{2\pi}.$$

We are done by $\frac{|g(a)|+|g(b)|}{2\pi} + \frac{|g(a)|-|g(b)|}{2\pi} = \frac{|g(a)|}{\pi}$. □

**Lemma A.4.** *Let $\sigma \geq 0$, $\tau \in \mathbb{R}$, $\nu \in \mathbb{R} \setminus \{0\}$. Let $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then, for any $k \geq 1$, $f(t) = t^{-\sigma-k}|2\pi\nu - \tau/t|^{-k-1}$ is decreasing on $[a,b]$.*

*Proof.* Let $a \leq t \leq b$. Since $\left|\frac{\tau}{t\nu}\right| < 2\pi$, we see that $2\pi - \frac{\tau}{\nu t} > 0$, and so $|2\pi\nu - \tau/t|^{-k-1} = |\nu|^{-k-1}\left(2\pi - \frac{\tau}{t\nu}\right)^{-k-1}$. Now we take logarithmic derivatives:

$$t(\log f(t))' = -(\sigma+k) - (k+1)\frac{\tau/t}{2\pi\nu - \tau/t} = -\sigma - \frac{2\pi k + \frac{\tau}{t\nu}}{2\pi - \frac{\tau}{t\nu}} < -\sigma \leq 0,$$

since, again by $\frac{|\tau|}{t|\nu|} < 2\pi$ and $k \geq 1$, we have $2\pi k + \frac{\tau}{t\nu} > 0$, and, as we said, $2\pi - \frac{\tau}{t\nu} > 0$. □

**Lemma A.5.** *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R} \setminus \{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then*

$$\int_a^b t^{-s} e(\nu t) dt = \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b + \frac{a^{-\sigma-1}}{2\pi^2} O^*\left(\frac{\sigma}{(\nu-\vartheta)^2} + \frac{|\vartheta|}{|\nu-\vartheta|^3}\right),$$

*where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$ and $\vartheta = \frac{\tau}{2\pi a}$.*

*Proof.* Apply Lemma A.1. Since $\varphi_\nu'(t) = \nu - \tau/(2\pi t)$, we know by Lemma A.4 (with $k = 1$) that $g_1(t) = \frac{t^{-\sigma-1}}{(\varphi_\nu'(t))^2}$ is decreasing on $[a, b]$. We know that $\varphi_\nu'(t) \neq 0$ for $t \geq a$ by $a > \frac{|\tau|}{2\pi|\nu|}$, and so we also know that $g_1(t)$ is continuous for $t \geq a$. Hence, by Lemma A.3,

$$\left|\int_a^b \frac{t^{-\sigma-1}}{2\pi i \varphi_\nu'(t)} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi} \cdot \frac{|g_1(a)|}{\pi} = \frac{1}{2\pi^2} \frac{a^{-\sigma-1}}{|\nu-\vartheta|^2},$$

since $\varphi_\nu'(a) = \nu - \vartheta$. We remember to include the factor of $\sigma$ in front of an integral in (A.1).

Since $\varphi_\nu'(t)$ is as above and $\varphi_\nu''(t) = \tau/(2\pi t^2)$, we know by Lemma A.4 (with $k = 2$) that $g_2(t) = \frac{t^{-\sigma}|\varphi_\nu''(t)|}{|\varphi_\nu'(t)|^3} = \frac{|\tau|}{2\pi} \frac{t^{-\sigma-2}}{|\varphi_\nu'(t)|^3}$ is decreasing on $[a, b]$; we also know, as before, that $g_2(t)$ is continuous. Hence, again by Lemma A.3,

$$\left|\int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i (\varphi_\nu'(t))^2} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi} \frac{|g_2(a)|}{\pi} = \frac{1}{2\pi^2} \frac{a^{-\sigma-1}|\vartheta|}{|\nu-\vartheta|^3}.$$

□

**Lemma A.6.** *Let $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. Let $n \in \mathbb{Z}_{>0}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi n}$. Write $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$. Then*

$$\frac{1}{2}\sum_{\nu=\pm n} \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b = \left.\frac{(-1)^n t^{-s} \cdot \frac{\tau}{2\pi t}}{2\pi i \left(n^2 - \left(\frac{\tau}{2\pi t}\right)^2\right)}\right|_a^b.$$

It is this easy step that gives us quadratic decay on $n$. It is just as in the proof of van der Corput's Process B in, say, [Ten15, I.6.3, Thm. 4].

*Proof.* Since $e(\varphi_\nu(t)) = e(\nu t) t^{-i\tau} = (-1)^\nu t^{-i\tau}$ for any half-integer $t$ and any integer $\nu$,

$$\left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b = \left.\frac{(-1)^\nu t^{-s}}{2\pi i \varphi_\nu'(t)}\right|_a^b$$

for $\nu = \pm n$. Clearly $(-1)^\nu = (-1)^n$. Since $\varphi_\nu'(t) = \nu - \alpha$ for $\alpha = \frac{\tau}{2\pi t}$,

$$\frac{1}{2}\sum_{\nu=\pm n} \frac{1}{\varphi_\nu'(t)} = \frac{1/2}{n-\alpha} + \frac{1/2}{-n-\alpha} = \frac{-\alpha}{\alpha^2 - n^2} = \frac{\alpha}{n^2 - \alpha^2}.$$

□

**Proposition A.7.** *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi}$. Write $\vartheta = \frac{\tau}{2\pi a}$. Then, for any integer $n \geq 1$,*

$$\int_a^b t^{-s}\cos 2\pi n t \, dt = \left.\left(\frac{(-1)^n t^{-s}}{2\pi i} \cdot \frac{\frac{\tau}{2\pi t}}{n^2 - \left(\frac{\tau}{2\pi t}\right)^2}\right)\right|_a^b + \frac{a^{-\sigma-1}}{4\pi^2} O^*\left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} + \frac{|\vartheta|}{|n-\vartheta|^3} + \frac{|\vartheta|}{|n+\vartheta|^3}\right).$$

*Proof.* Write $\cos 2\pi nt = \frac{1}{2}(e(nt) + e(-nt))$. Since $n \geq 1$ and $a > \frac{|\tau|}{2\pi}$, we know that $a > \frac{|\tau|}{2\pi n}$, and so we can apply Lemma A.5 with $\nu = \pm n$. We then apply Lemma A.6 to combine the boundary contributions $|_a^b$ for $\nu = \pm n$. $\square$

*Notation.* We recall that $e(\alpha) = e^{2\pi i\alpha}$, and $O^*(R)$ means a quantity of absolute value at most $R$. We write $\sum_n$ for $\sum_{n\geq 1}$, $\sum_{n\leq a}$ for $\sum_{1\leq n\leq a}$, etc. By $\{x\}$ we denote the fractional part $x - \lfloor x\rfloor$ of $x$. We define the Fourier transform

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e(-xt)dt$$

for $f \in L^1(\mathbb{R})$.

A.2. **Approximating $\zeta(s)$.**

**Lemma A.8.** *Let $b > 0$, $b \in \mathbb{Z} + \frac{1}{2}$. Then, for all $s \in \mathbb{C} \setminus \{1\}$ with $\Re s > 0$,*

$$\sum_{n\leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + s\int_b^\infty \left(\{y\} - \frac{1}{2}\right)\frac{dy}{y^{s+1}}. \tag{A.2}$$

*Proof.* Assume first that $\Re s > 1$. By first-order Euler-Maclaurin and $b \in \mathbb{Z} + \frac{1}{2}$,

$$\sum_{n>b} \frac{1}{n^s} = \int_b^\infty \frac{dy}{y^s} + \int_b^\infty \left(\{y\} - \frac{1}{2}\right) d\left(\frac{1}{y^s}\right).$$

Here $\int_b^\infty \frac{dy}{y^s} = -\frac{b^{1-s}}{1-s}$ and $d\left(\frac{1}{y^s}\right) = -\frac{s}{y^{s+1}}dy$. Hence, by $\sum_{n\leq b}\frac{1}{n^s} = \zeta(s) - \sum_{n>b}\frac{1}{n^s}$ for $\Re s > 1$,

$$\sum_{n\leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + \int_b^\infty \left(\{y\} - \frac{1}{2}\right)\frac{s}{y^{s+1}}dy.$$

Since the integral converges absolutely for $\Re s > 0$, both sides extend holomorphically to $\{s \in \mathbb{C} : \Re s > 0, s \neq 1\}$; thus, the equation holds throughout that region. $\square$

**Corollary A.9.** *Let $b > a > 0$, $b \in \mathbb{Z} + \frac{1}{2}$. Then, for all $s \in \mathbb{C} \setminus \{1\}$ with $\sigma = \Re s > 0$,*

$$\sum_{n\leq a} \frac{1}{n^s} = -\sum_{a<n\leq b}\frac{1}{n^s} + \zeta(s) + \frac{b^{1-s}}{1-s} + O^*\left(\frac{|s|}{2\sigma b^\sigma}\right).$$

*Proof.* By Lemma A.8, $\sum_{n\leq a} = \sum_{n\leq b} - \sum_{a<n\leq b}$, $\left|\{y\} - \frac{1}{2}\right| \leq \frac{1}{2}$ and $\int_b^\infty \frac{dy}{|y^{s+1}|} = \frac{1}{\sigma b^\sigma}$. $\square$

**Lemma A.10.** *Let $a, b \in \mathbb{R} \setminus \mathbb{Z}$, $b > a > 0$. Let $s \in \mathbb{C} \setminus \{1\}$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(y) = 1_{[a,b]}(y)/y^s$. Then*

$$\sum_{a<n\leq b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} + \lim_{N\to\infty}\sum_{n=1}^N (\widehat{f}(n) + \widehat{f}(-n)).$$

*Proof.* Since $a \notin \mathbb{Z}$, $\sum_{a<n\leq b}\frac{1}{n^s} = \sum_{n\in\mathbb{Z}} f(n)$. By Poisson summation (as in [MV07, Thm. D.3])

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{N\to\infty}\sum_{n=-N}^N \widehat{f}(n) = \widehat{f}(0) + \lim_{N\to\infty}\sum_{n=1}^N(\widehat{f}(n) + \widehat{f}(-n)),$$

where we use the facts that $f$ is in $L^1$, of bounded variation, and (by $a, b \notin \mathbb{Z}$) continuous at every integer. Now

$$\widehat{f}(0) = \int_{\mathbb{R}} f(y)dy = \int_a^b \frac{dy}{y^s} = \frac{b^{1-s} - a^{1-s}}{1-s}.$$

$\square$

**Lemma A.11** (Euler/Mittag-Leffler expansion for csc and $\csc^2$)**.** *Let $z \in \mathbb{C}$, $z \notin \mathbb{Z}$. Then*

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_n (-1)^n \left(\frac{1}{z-n} + \frac{1}{z+n}\right), \quad \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

We could prove these equations starting from Euler's product for $\sin \pi z$.

*Proof.* Let us start from the Mittag-Leffler expansion $\pi \cot \pi s = \frac{1}{s} + \sum_n \left(\frac{1}{s-n} + \frac{1}{s+n}\right)$.

Applying the trigonometric identity $\cot u - \cot\left(u + \frac{\pi}{2}\right) = \cot u + \tan u = \frac{2}{\sin 2u}$ with $u = \pi z/2$, and letting $s = z/2$, $s = (z+1)/2$, we see that

$$\begin{aligned}
\frac{\pi}{\sin \pi z} &= \frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z+1)}{2} \\
&= \frac{1/2}{z/2} + \sum_n \left(\frac{1/2}{\frac{z}{2} - n} + \frac{1/2}{\frac{z}{2} + n}\right) - \frac{1/2}{(z+1)/2} - \sum_n \left(\frac{1/2}{\frac{z+1}{2} - n} + \frac{1/2}{\frac{z+1}{2} + n}\right) \\
&= \frac{1}{z} + \sum_n \left(\frac{1}{z-2n} + \frac{1}{z+2n}\right) - \sum_n \left(\frac{1}{z-(2n-1)} + \frac{1}{z+(2n-1)}\right)
\end{aligned}$$

after reindexing the second sum. Collecting the odd and even terms, we obtain our first equation.

We may differentiate the expansion of $\pi \cot \pi z$ term-by-term because it converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$. By $(\pi \cot \pi z)' = -\frac{\pi^2}{\sin^2 \pi z}$ and $\left(\frac{1}{z \pm n}\right)' = -\frac{1}{(z \pm n)^2}$, we obtain our second equation. □

**Lemma A.12.** *For $\vartheta \in \mathbb{R}$ with $0 \leq |\vartheta| < 1$,*

$$\sum_n \left(\frac{1}{(n-\vartheta)^3} + \frac{1}{(n+\vartheta)^3}\right) \leq \frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1.$$

*Proof.* Since $\frac{1}{(n-\vartheta)^3} + \frac{1}{(n+\vartheta)^3}$ is even, we may replace $\vartheta$ by $|\vartheta|$. Then we rearrange the sum:

$$\sum_{n=1}^{\infty} \left(\frac{1}{(n-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3}\right) = \frac{1}{(1-|\vartheta|)^3} + \sum_{n=1}^{\infty} \left(\frac{1}{(n+1-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3}\right).$$

We may write $(n+1-|\vartheta|)^3$, $(n+|\vartheta|)^3$ as $(n+\frac{1}{2}-t)^3$, $(n+\frac{1}{2}+t)^3$ for $t = |\vartheta| - 1/2$. Since $1/u^3$ is convex, $\frac{1}{(n+1/2-t)^3} + \frac{1}{(n+1/2+t)^3}$ reaches its maximum on $[-1/2, 1/2]$ at the endpoints. Hence

$$\sum_{n=1}^{\infty} \left(\frac{1}{(n+1-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3}\right) \leq \sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{(n+1)^3}\right) = 2\zeta(3) - 1.$$

□

**Lemma A.13.** *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$, with $s \neq 1$. Let $b > a > 0$, $a, b \in \mathbb{Z} + \frac{1}{2}$, with $a > \frac{|\tau|}{2\pi}$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(y) = 1_{[a,b]}(y)/y^s$. Write $\vartheta = \frac{\tau}{2\pi a}$, $\vartheta_- = \frac{\tau}{2\pi b}$. Then*

$$\sum_n (\widehat{f}(n) + \widehat{f}(-n)) = \frac{a^{-s} g(\vartheta)}{2i} - \frac{b^{-s} g(\vartheta_-)}{2i} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}}\right)$$

*with absolute convergence, where $g(t) = \frac{1}{\sin \pi t} - \frac{1}{\pi t}$ for $t \neq 0$, $g(0) = 0$, and*

$$C_{\sigma,\vartheta} = \begin{cases} \frac{\sigma}{2}\left(\frac{1}{\sin^2 \pi\vartheta} - \frac{1}{(\pi\vartheta)^2}\right) + \frac{|\vartheta|}{2\pi^2}\left(\frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1\right) & \text{for } \vartheta \neq 0, \\ \sigma/6 & \text{for } \vartheta = 0. \end{cases} \tag{A.3}$$

*Proof.* By Proposition A.7, multiplying by 2 (since $e(-nt) + e(nt) = 2\cos 2\pi nt$),

$$\begin{aligned}\widehat{f}(n) + \widehat{f}(-n) &= \frac{a^{-s}}{2\pi i}\frac{(-1)^{n+1} 2\vartheta}{n^2 - \vartheta^2} - \frac{b^{-s}}{2\pi i}\frac{(-1)^{n+1} 2\vartheta_-}{n^2 - \vartheta_-^2} \\ &\quad + \frac{a^{-\sigma-1}}{2\pi^2} O^*\left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} + \frac{|\vartheta|}{(n-\vartheta)^3} + \frac{|\vartheta|}{(n+\vartheta)^3}\right),\end{aligned} \tag{A.4}$$

where $\vartheta_- = \tau/(2\pi b)$. By the first equation in Lemma A.11,

$$\sum_n \frac{(-1)^{n+1} 2z}{n^2 - z^2} = \frac{\pi}{\sin \pi z} - \frac{1}{z}$$

for $z \neq 0$, while $\sum_n \frac{(-1)^{n+1}2z}{n^2 - z^2} = \sum_n 0 = 0$ for $z = 0$.

Moreover, by Lemmas A.11 and A.12, for $\vartheta \neq 0$,

$$\sum_n \left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2}\right) \leq \sigma \cdot \left(\frac{\pi^2}{\sin^2 \pi\vartheta} - \frac{1}{\vartheta^2}\right),$$

$$\sum_n \left(\frac{|\vartheta|}{(n-\vartheta)^3} + \frac{|\vartheta|}{(n+\vartheta)^3}\right) \leq |\vartheta| \cdot \left(\frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1\right).$$

If $\vartheta = 0$, then $\sum_n \left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2}\right) = 2\sigma \sum_n \frac{1}{n^2} = \frac{\sigma\pi^2}{3}$, and then the quantity in (A.4) is bounded by $\frac{a^{-\sigma-1}}{2\pi^2} \cdot \frac{\sigma\pi^2}{3} = \frac{\sigma/6}{a^{\sigma+1}}$. □

**Proposition A.14.** *Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$, with $s \neq 1$. Let $a \in \mathbb{Z} + \frac{1}{2}$ with $a > \frac{|\tau|}{2\pi}$. Then*

$$\zeta(s) = \sum_{n \leq a} \frac{1}{n^s} - \frac{a^{1-s}}{1-s} + c_\vartheta a^{-s} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}}\right), \tag{A.5}$$

*where $\vartheta = \frac{\tau}{2\pi a}$, $c_\vartheta = \frac{i}{2}\left(\frac{1}{\sin \pi\vartheta} - \frac{1}{\pi\vartheta}\right)$ for $\vartheta \neq 0$, $c_0 = 0$, and $C_{\sigma,\vartheta}$ is as in* (A.3).

If $0 \leq \sigma < 1$, then $\frac{a^{1-s}}{1-s} = \int_0^a \frac{dy}{y^s}$, and so (A.5) can be read as expressing a difference $\sum_{n \leq a} - \int_0^a$ as a constant $\zeta(s)$ plus an error term.

*Proof.* Assume first that $\sigma > 0$. Let $b \in \mathbb{Z} + \frac{1}{2}$ with $b > a$, and define $f(y) = \frac{1_{[a,b]}(y)}{y^s}$. By Corollary A.9 and Lemma A.10,

$$\sum_{n \leq a} \frac{1}{n^s} = \zeta(s) + \frac{a^{1-s}}{1-s} - \lim_{N \to \infty} \sum_{n=1}^N (\widehat{f}(n) + \widehat{f}(-n)) + O^*\left(\frac{2|s|}{\sigma b^\sigma}\right).$$

We apply Lemma A.13 to estimate $\lim_{N\to\infty} \sum_{n=1}^N (\widehat{f}(n) + \widehat{f}(-n))$. We obtain

$$\sum_{n \leq a} \frac{1}{n^s} = \zeta(s) + \frac{a^{1-s}}{1-s} - \frac{a^{-s} g(\vartheta)}{2i} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}}\right) + \frac{b^{-s} g(\vartheta_-)}{2i} + O^*\left(\frac{2|s|}{\sigma b^\sigma}\right),$$

where $\vartheta_- = \frac{\tau}{2\pi b}$ and $g(t)$ is as in Lemma A.13, and so $-\frac{g(\vartheta)}{2i} = c_\vartheta$. We let $b \to \infty$ through the half-integers, and obtain (A.5), since $b^{-\sigma} \to 0$, $\vartheta_- \to 0$ and $g(\vartheta_-) \to g(0) = 0$ as $b \to \infty$.

Finally, the case $\sigma = 0$ follows since all terms in (A.5) extend continuously to $\sigma = 0$. □

*Remark.* The term $c_\vartheta a^{-s}$ in (A.5) does not seem to have been worked out before in the literature; the factor of $i$ in $c_\vartheta$ was a surprise. For the sake of comparison, let us note that, if $a \geq x$, then $|\vartheta| \leq 1/2\pi$, and so $|c_\vartheta| \leq |c_{\pm 1/2\pi}| = 0.04291\ldots$ and $|C_{\sigma,\vartheta}| \leq |C_{\sigma,\pm 1/2\pi}| \leq 0.176\sigma + 0.246$. While $c_\vartheta$ is optimal, $C_{\sigma,\vartheta}$ need not be – but then that is irrelevant for most applications.

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