# Towards a formal proof of the Freyd-Mitchell embedding theorem

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July 25, 2024

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D'
\end{tikzcd}$$

# The embedding theorem

**Theorem (Freyd-Mitchell 1964)**

*Let $\mathcal{C}$ be a small abelian category. Then there is a (not necessarily commutative) ring $R$ and a full, faithful and exact functor $\mathcal{C} \to \mathrm{Mod}_R$.*

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- "Diagram chases over modules are valid in all abelian categories"
- Can even define morphisms this way!

# Is this useful? (1)

Peter Johnstone (1977):

> *Incidentally, the Freyd-Mitchell embedding theorem is frequently regarded as a culmination rather than a starting-point; this is because of what seems to me a misinterpretation (or at least an inversion) of its true significance.*
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> *[...] I believe its true import is "If you want to prove something about categories of modules, you might as well work in a general abelian category"—for the embedding theorem ensures that your result will be true in this generality, and by forgetting the explicit structure of module categories you will be forced to concentrate on the essential aspects of the problem.*

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Formalization experience:

> *It might not be strictly necessary, but it's certainly useful!*

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- Pseudoelements (Mac Lane 1971, Borceux 1994, mathlib: 2020)
  - Associate to an object $X$ a set of *pseudoelements*, which are equivalence classes of morphisms $\bullet \to X$.
  - $X \to Y$ induces $\mathrm{PseudoElem}(X) \to \mathrm{PseudoElem}(Y)$
  - $X \to Y$ epi iff $\mathrm{PseudoElem}(X) \to \mathrm{PseudoElem}(Y)$ surjective, etc.

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```
mono_of_zero_of_map_zero _ fun c hc =>
  have : h c = 0 :=
    suffices δ (h c) = 0 from zero_of_map_zero _ (pseudo_injective_of_mono _) _ this
    calc
      δ (h c) = h' (γ c) := by rw [← Pseudoelement.comp_apply, ← comm₃, Pseudoelement.comp_apply]
      _ = h' 0 := by rw [hc]
      _ = 0 := apply_zero _
  Exists.elim ((pseudo_exact_of_exact hgh).2 _ this) fun b hb =>
    have : g' (β b) = 0 :=
      calc
        g' (β b) = γ (g b) := by rw [← Pseudoelement.comp_apply, comm₂, Pseudoelement.comp_apply]
        _ = γ c := by rw [hb]
        _ = 0 := hc
    Exists.elim ((pseudo_exact_of_exact hf'g').2 _ this) fun a' ha' =>
      Exists.elim (pseudo_surjective_of_epi α a') fun a ha =>
        have : f a = b :=
          suffices β (f a) = β b from pseudo_injective_of_mono _ this
          calc
```

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```
apply mono_of_cancel_zero
intro A f₂ h₁
have h₂ : f₂ ≫ R₁.map' 2 3 = 0 := by
  rw [← cancel_mono (app' φ 3 _), assoc, NatTrans.naturality, reassoc_of% h₁,
    zero_comp, zero_comp]
obtain ⟨A₁, π₁, _, f₁, hf₁⟩ := (hR₁'.exact 0).exact_up_to_refinements f₂ h₂
dsimp at hf₁
have h₃ : (f₁ ≫ app' φ 1) ≫ R₂.map' 1 2 = 0 := by
  rw [assoc, ← NatTrans.naturality, ← reassoc_of% hf₁, h₁, comp_zero]
obtain ⟨A₂, π₂, _, g₀, hg₀⟩ := (hR₂.exact 0).exact_up_to_refinements _ h₃
obtain ⟨A₃, π₃, _, f₀, hf₀⟩ := surjective_up_to_refinements_of_epi (app' φ 0 _) g₀
have h₄ : f₀ ≫ R₁.map' 0 1 = π₃ ≫ π₂ ≫ f₁ := by
  rw [← cancel_mono (app' φ 1 _), assoc, assoc, assoc, NatTrans.naturality,
    ← reassoc_of% hf₀, hg₀]
```

# How to prove the embedding theorem (1)

**Theorem**

*Let $\mathcal{C}$ be a small abelian category. Then there is a (not necessarily commutative) ring $R$ and a full, faithful and exact functor $\mathcal{C} \to \mathrm{Mod}_R$.*

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Suppose we have chosen an object $P$ of $\mathcal{C}$. Set $R \coloneqq \mathrm{Hom}(P, P)$. Get

$$\begin{aligned} Y(P) : \mathcal{C} &\to \mathrm{Mod}_R \\ X &\mapsto \mathrm{Hom}(P, X) \end{aligned}$$

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Suppose we have chosen an object $P$ of $\mathcal{C}$. Set $R := \mathrm{Hom}(P, P)$. Get

$$\begin{aligned} Y(P) : \mathcal{C} &\to \mathrm{Mod}_R \\ X &\mapsto \mathrm{Hom}(P, X) \end{aligned}$$

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This is

- always left exact,
- right exact iff $P$ is projective,
- faithful iff $P$ is a generator,
- full as long as all of the above hold and $\mathcal{C}$ is cocomplete.

Not every small abelian category is cocomplete and has a projective generator!

# How to prove the embedding theorem (2)

**Theorem**

*Let $\mathcal{C}$ be a small abelian category. Then there is a (not necessarily commutative) ring $R$ and a full, faithful and exact functor $\mathcal{C} \to \mathsf{Mod}_R$.*

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*Let $\mathcal{C}$ be a small abelian category. Then there is a (not necessarily commutative) ring $R$ and a full, faithful and exact functor $\mathcal{C} \to \mathrm{Mod}_R$.*

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- Know how to do it if $\mathcal{C}$ is cocomplete and has a projective generator, what about the general case?
- Idea: embed $\mathcal{C}$ into a category that has these properties.
- Freyd uses $Y : \mathcal{C} \to \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.
- Kashiwara-Schapira (and we) use $Y : \mathcal{C} \to \mathrm{Pro}(\mathcal{C}) \cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.
- $\mathrm{Pro}(\mathcal{C})$ works because it is a co-Grothendieck category.

# How to prove the embedding theorem (3)

Rough list of things to do to prove the theorem:

- Define Grothendieck categories (easy enough)
- Show that Grothendieck categories are complete (SAFT, done)
- Show that Grothendieck categories have an injective cogenerator (hard)
- Define $\mathrm{Pro}(\mathcal{C})$ and prove basic properties (hard, very long, working on it)
- Show that $\mathrm{Pro}(\mathcal{C})$ is a co-Grothendieck category (hopefully doable)
- Prove various properties of the two embeddings $\mathcal{C} \to \mathrm{Pro}(\mathcal{C}) \to \mathrm{Mod}_R$ (easy but somewhat long, mostly done)

Other proofs are possible.

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- The proof is very very long
- We work on this in our free time on weekends
- Project started in February 2022, will realistically take several more years
- How to combat "quadratic bitrot"?
- Solution: use a proof that develops a lot of general theory and PR everything to mathlib immediately
- 138 PRs merged so far (97 to mathlib3, 61 to mathlib4)
- Amount of unmerged code: $< 300$ lines
- Additional benefits:
  - Do things right from the start, no accumulating piles of hacks (but cannot take any shortcuts)
  - Partial progress is useful to other people

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Every category is small if you look at it from far enough away. . .

# Size issues (2)

**Theorem (Yoneda 1954)**

*Let $\mathcal{C}$ be a category, let $F : \mathcal{C} \to \mathrm{Set}$ be a functor and let $X$ be an object of $\mathcal{C}$. Then there is a bijection between the natural transformations $\mathrm{Hom}(X, -) \to F$ and $FX$.*

# Is this useful? (2)

Advantage when defining data is much less clear.