Lusin Novikov in Lean

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1 Completed Lemmas

Throughout, suppose X and Y are topological spaces, and $f : X \to Y$ is continuous. All families of sets are assumed to be pairwise disjoint. Let I be the sigma ideal generated by Borel partial sections of f (in particular, $A \in I$ if and only if $A = \bigcup_{n\in\mathbb{N}} A_n$, where each A_n is Borel and $f \upharpoonright A_n$ is injective for all n). We say a family $\mathcal{F}$ of sets is *null* if there exists a Borel cover $(B_F)_{F\in\mathcal{F}}$ of Y such that for each $F \in \mathcal{F}$, $F \cap f^{-1}(B_F) \in I$. One can show that a singleton $\{F\}$ is null if and only if $F \in I$.

Lemma 1.1. *Suppose $\mathcal{F} \cup \{U\}$ is a family of sets in X with U Borel, and further suppose we can write $(U)^2 = \bigcup_{n\in\mathbb{N}} V_n \times W_n$ with all V_n, W_n Borel and each $\mathcal{F} \cup \{V_n, W_n\}$ null. Then $F \cup \{U\}$ is null.*

Lemma 1.2. *Suppose $\mathcal{F}$ is a family of sets in X, and suppose there is a Borel cover $(A_n)_{n\in\mathbb{N}}$ of Y such that for each $n \in \mathbb{N}$, the family $(F \cap f^{-1}(A_n))_{F\in\mathcal{F}}$ is null. Then $\mathcal{F}$ is null.*

Observation 1. *Suppose X is separable and metrizable, and fix a metric on X. Given $\varepsilon > 0$ and closed $U \subseteq X$, we can write $(U)^2 = \bigcup_{n\in\mathbb{N}} V_n \times W_n$ for closed $V_n, W_n \subseteq U$ of diameter $\leq \varepsilon$.*

Observation 2. *Suppose Y is separable and metrizable, and fix a metric on Y. Given $\varepsilon > 0$, we can write Y as a countable union of closed sets of diameter $\leq \varepsilon$.*

2 To Be Proved

Lemma 2.1 (Splitting one set)**.** *Suppose X is separable and metrizable, and a metric on it. Further suppose $\mathcal{F} = \{F_1, \dots, F_m\}$ is a non-null family of closed sets in X. Then for any $\varepsilon > 0$ and $k \leq m$ there exist closed disjoint $F_{k,0}, F_{k,1} \subseteq F_k$ of diameter $\leq \varepsilon$ such that $\{F_1, \dots, F_{k-1}, F_{k,0}, F_{k,1}, F_{k+1} \dots, F_m\}$ is non-null.*

Proof. By Observation 1, we can find closed $V_n, W_n \subseteq F_1$ of diameter $\leq \varepsilon$ such that $(F_1)^2 = \bigcup_{n\in\mathbb{N}} V_n \times W_n$. We claim there is some n such that $\{F_2, \dots, F_m\} \cup \{V_n, W_n\}$ is non-null. Indeed, if not, then by Lemma 1.1, $\{F_2, \dots, F_m\} \cup \{F_1\} = \mathcal{F}$ would be null, a contradiction. Fixing such an n, we may take $F_{1,0} = V_n$ and $F_{1,1} = W_n$, noting they are closed and of appropriate diameter for free, and they are disjoint as $U_1 \times U_2 \subseteq (U)^2$. □

Lemma 2.2 (Splitting multiple sets)**.** *Suppose X is separable and metrizable, and fix a metric on it. Further suppose $\mathcal{F} = \{F_1, \dots, F_m\}$ is a non-null family of closed sets in X and $k \leq m$. Then for any $\varepsilon > 0$, there exist closed disjoint $F_{1,0}, F_{1,1} \subseteq F_1, \dots, F_{k,0}, F_{k,1} \subseteq F_k$ of diameter $\leq \varepsilon$ such that $\{F_{1,0}, F_{1,1}, \dots, F_{k,0}, F_{k,1}, F_{k+1}, \dots, F_m\}$ is non-null.*

Proof. We proceed by induction on k. For the base case $k = 0$ there's nothing to do. Now suppose the result holds for all $k \leq i$, and consider $k = i + 1$. By Lemma 2.1, we can find closed disjoint $F_{i+1,0}, F_{i+1,1} \subseteq F_{i+1}$ of appropriate diameter such that $\{F_1, \dots, F_i, F_{i+1,0}, F_{i+1}, \dots, F_m\}$ is non-null. We can then apply the induction hypothesis to this collection to split the first i sets. □

Lemma 2.3 (Refining the image)**.** *Suppose Y is separable and metrizable, and fix a metric on it. Further suppose $\mathcal{F}$ is a non-null family of closed sets in X. Then for any $\varepsilon > 0$ there exists a non-null family $(S_F)_{F\in\mathcal{F}}$ of closed sets in X such that $S_F \subseteq F$ for all $F \in \mathcal{F}$ and $f(\bigcup_{F\in\mathcal{F}} S_F)$ has diameter $\leq \varepsilon$.*

Proof. By Observation 2, we can write $Y = \bigcup_{n\in\mathbb{N}} A_n$ for closed A_n of diameter $\le \varepsilon$. We claim there exists an n such that the family $(F \cap f^{-1}(A_n))_{F\in\mathcal{F}}$ is non-null. If not, by Lemma 1.2 $\mathcal{F}$ is null, a contradiction. Fixing such an n, set $S_F = F \cap f^{-1}(A_n) \subseteq F$. Each S_F is closed since since F and A_n are and f is continuous. Furthermore, $f(\bigcup_{F\in\mathcal{F}}) \subseteq A_n$, which has diameter $\le \varepsilon$ by construction, so we're done. □

Lemma 2.4 (Splitting Lemma)**.** *Suppose X and Y are separable and metrizable, and fix metrics on both of them. Further suppose $\mathcal{F} = F_1, \dots, F_m$ is a non-null family of closed sets in X. Then for any $\varepsilon > 0$, there exist closed $F_{1,0}, F_{1,1} \subseteq F_1, \dots, F_{m,0}, F_{m,1} \subseteq F_m$ of diameter $\le \varepsilon$ such that $f(\bigcup_{n=1}^{m}\{F_{n,1}, F_{n,2}\})$ has diameter $\le \varepsilon$ and $\bigcup_{n=1}^{m}\{F_{n,1}, F_{n,2}\}$ is non-null.*

Proof. By Lemma 2.2 with $k = m$, we may obtain closed disjoint $F'_{n,0}, F'_{n,1} \subseteq F_n$ for each i with diameter $\le \varepsilon$ such that $\bigcup_{n=1}^{m}\{F'_{n,0}, F'_{n,1}\}$ is non-null. Applying Lemma 2.3 to the collection of F's, we obtain the desired collection. □

Lusin-Novikov Theorem. *Let X be a Polish space and Y be a separable metrizable space. Exactly one of the following holds.*

(1) *X can be covered by countably many Borel partial sections of f.*

(2) *A fiber of f contains a Cantor set.*

Proof. Note that (1) and (2) cannot both simultaneously hold, as (2) implies you need at least continuum-many partial sections to cover X. It thus suffices to show $\neg(1) \Rightarrow (2)$. Towards that end, suppose X cannot be covered by countably many Borel partial sections of f. This means $X \notin I$ and thus $\{X\}$ is non-null.

For each $n \in \mathbb{N}$ we define a non-null collection (F_ℓ) of closed disjoint sets in X indexed by $0, 1$-strings of ℓ length n by recursion on n. For $n = 0$, define $F_{\text{nil}} = X$. For $n + 1$, we apply the splitting lemma with $\varepsilon = 2^{-n}$ on the collection for length n, and define $F_{\ell^\frown 0} = F_{\ell,0}$ and $F_{\ell^\frown 1} = F_{\ell,1}$ for each ℓ.

Using the induced map from the CantorScheme library, we can show the scheme is ClosureAntitone, disjoint, and has vanishing diameter, so the induced map is total, continuous, and injective, thus getting an injection from the Cantor set into X. We can then use the vanishing diameter on the output to see that this entire Cantor set lies in a single fiber of f. □