Lean for PDEs - Differential calculus

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Our goal

A possible formalization of a general PDE

```
structure IsClassicalSolution
    (F : FormalMultilinearSeries 𝕜 E G → E → G)
    (n : ℕ) (U : Set E) (u : E → G) : Prop where
  condDiffOn : ContDiffOn 𝕜 n u U
  eq_zero : ∀ x ∈ U, F (ftaylorSeriesWithin 𝕜 u U x) x = 0
```

$u\colon E \to G$ is n times continuously differentiable on $U \subseteq E$ and satisfies the PDE $F(\ldots, D^k u(x), \ldots, Du(x), u(x), x) = 0$ for all $x \in U$.

This definition is not from Mathlib. But it uses only definitions from Mathlib.

Normed spaces

Let E be a normed space over $\mathbb{R}$.

```
variable {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
```

Banach space: `[CompleteSpace E]`.
Finite dimension: `[FiniteDimensional ℝ E]` (complete space is automatically inferred).

See Mathematics in Lean book, chapter 12.

Continuous linear maps

Let E and F be normed spaces over $\mathbb{R}$.

The derivative of a function $f : E \to F$ at a point $(x : E)$ is a continuous linear map from E to F.
Continuous linear maps: `f' : E →L[ℝ] F`, notation for the `ContinuousLinearMap` type

The derivative of f at x is f': `HasFDerivAt f f' x` . The F stands for Fréchet.

```
HasFDerivAt f f' x₀ ↔ (fun x ↦ f x - f x₀ - f' (x - x₀)) =o[𝓝 x₀] fun x ↦ x - x₀
```

f has a derivative at x: `DifferentiableAt f x`.

```
DifferentiableAt ℝ f x = ∃ (f' : E →L[ℝ] F), HasFDerivAt f f' x
```

Derivatives within, on, and everywhere

- At: when the property is true at the point x.
- Within: when we approach x only from within the set s.
- On: when the property is true "within" for all $x \in s$.

```
HasFDerivWithinAt
DifferentiableWithinAt ℝ f s x = ∃ f′ : E →L[ℝ] F, HasFDerivWithinAt f f′ s x
DifferentiableOn ℝ f s = ∀ x ∈ s, DifferentiableWithinAt ℝ f s x
Differentiable ℝ f = ∀ x, DifferentiableAt ℝ f x
```

HasFDeriv vs fderiv

`HasFDerivAt f f' x` means that f' is the derivative of f at x.

To obtain the derivative of f at x, use `fderiv ℝ f x`.
Note: defined even if f is not differentiable at x (with the default value 0).

Use `DifferentiableAt ℝ f x` as hypothesis to ensure that `fderiv ℝ f x` is meaningful.

Note: prove differentiability with `fun_prop` (tactic for function properties).

Higher derivatives

The n-th derivative of $f : E \to G$ at $(x : E)$ is a continuous n-multilinear map from E^n to G.

`E[×n]→L[ℝ] G`, notation for the `ContinuousMultilinearMap` type

`FormalMultilinearSeries`: sequence of continuous n-multilinear maps, for $n \in \mathbb{N}$.

`HasFTaylorSeriesUpToOn n f p s` : f has a the formal multilinear series p as derivatives up to order n within s.

`iteratedFDerivWithin ℝ n f s x`: the n-th derivative of f at x within s. Defined with a default value if f is not differentiable enough.

`ftaylorSeriesWithin ℝ f s x`: the formal multilinear series of derivatives of f at x within s.

`ftaylorSeriesWithin ℝ f s x n = iteratedFDerivWithin ℝ n f s x`

Continuously differentiable functions

`ContDiffOn ℝ n f s`: *f* is *n* times continuously differentiable on *s*.
`n` takes values in `WithTop ℕ∞`, the natural numbers plus infinity, and an additional top element.

It means:

- *n* finite: *f* is C^n on *s*.
- *n* infinite: *f* is C^∞ on *s*.
- *n* equal to top: *f* is analytic on *s*.

`ContDiff ℝ n f`: *f* is *n* times continuously differentiable on the whole space.

New paper on this: Higher Order Differential Calculus in Mathlib, Sébastien Gouëzel

Generic PDE

Back to the tentative definition of a classical solution of a PDE.

```
structure IsClassicalSolution
    (F : FormalMultilinearSeries 𝕜 E G → E → G)
    (n : ℕ) (U : Set E) (u : E → G) : Prop where
  condDiffOn : ContDiffOn 𝕜 n u U
  eq_zero : ∀ x ∈ U, F (ftaylorSeriesWithin 𝕜 u U x) x = 0
```

- A PDE is a function F of a formal multilinear series and a point in E.
- A function u is a classical solution if it is n times continuously differentiable on U and F applied to the formal multilinear series of derivatives of u at x and x itself is zero for all $x \in U$.

Potentially missing: the statement that F is of order n.

Is this a useful definition? I don't know!