# 1 1323 $\rightarrow$ 2744 ?

We have an arbitrary magma $M$ equipped with a binary operator $\diamond$, such that the property **1323** holds:

$$x = y \diamond (((y \diamond y) \diamond x) \diamond y). \tag{1.1}$$

Does it then imply the property **2744**?:

$$x = ((y \diamond y) \diamond (y \diamond x)) \diamond y \tag{1.2}$$

We study the diagonal/squaring by setting it first constant $x \diamond x = c$ and include secondly an exception $w \diamond w \neq c$.

## 1.1 The diagonal/squaring $y \diamond y$ is not constant

Assume $y \diamond y = c$ for all $y \in M$ and an arbitrary $c \in M$. Now

$$x = y \diamond ((c \diamond x) \diamond y). \tag{1.3}$$

Set $y = c \diamond x$. Now

$$x = (c \diamond x) \diamond c \tag{1.4}$$

and therefore by setting $y = c$ in (1.3) and using (1.4):

$$x = c \diamond ((c \diamond x) \diamond c) = c \diamond x \tag{1.5}$$

which also immediately yields $x = x \diamond c$. Now property 1323 becomes

$$x = y \diamond (x \diamond y). \tag{1.6}$$

and property 2744

$$x = (y \diamond x) \diamond y. \tag{1.7}$$

which are equivalent ("Putnam Law"). Hence if $y \diamond y$ is constant, no counterexample can be derived.

### 1.1.1 Relaxation $c \diamond c \neq c$

We set $c \diamond c = c^2$. Now:

$$c^2 = c \diamond (((c \diamond c) \diamond c^2) \diamond c) = c \diamond c^2. \tag{1.8}$$

Only $x = c^2$ fullfills $c^2 \diamond x = c$:

$$x = c \diamond (((c \diamond c) \diamond x) \diamond c) = c \diamond c^2 = c^2. \tag{1.9}$$

Additionally:

$$c = c^2 \diamond (((c^2 \diamond c^2) \diamond c) \diamond c^2) = c^2 \diamond c. \tag{1.10}$$

Hence by (1.9) $c = c^2$, which goes against the definition of $c$ and $c^2$.

### 1.1.2 Relaxation $\exists! w : w \diamond w = w^2 \neq c,\ w \neq c$

Assume $c \diamond w \neq w$. By inserting $x = w$, $y = c \diamond w$ in (1.1) follows:

$$w = (c \diamond w) \diamond c \tag{1.11}$$

and by using this result und setting $x = w$, $y = c$ in (1.1) one derives

$$w = c \diamond (((c \diamond c) \diamond w) \diamond c) = c \diamond w. \tag{1.12}$$

But this is a contradiction. Hence $c \diamond w = w$. Also $x = w$ is the only element of the magma such that $c \diamond x = w$:

$$x = c \diamond (((c \diamond c) \diamond x) \diamond c) = c \diamond (((c \diamond c) \diamond w) \diamond c) = w. \tag{1.13}$$

Therefore for all others $x \neq w$ holds $c \diamond x \neq w$ and with the same analysis as in section 1.1 and our knowledge about $w$ immediately $c \diamond x = x$ follows for **all** $x \in M$. By inserting $x = w$ in (1.13) follows by this uniqueness

$$w = (c \diamond w) \diamond c = w \diamond c. \tag{1.14}$$

The same holds in general: $x = x \diamond c$ for all $x \in M$.
Additionally if $w^2 = w$, then $w^2 \diamond w = w^2 \diamond c$ and with (1.1) one derives $w = c$, a contradiction. Hence $w^2 \neq w$. Additionally

$$c = w \diamond ((w^2 \diamond c) \diamond w) = w \diamond (w^2 \diamond w) \tag{1.15}$$

hence $w^2 \diamond w \neq w$. And also $w \diamond w^2 \neq w$, since

$$w = w^2 \diamond (w \diamond w^2) \tag{1.16}$$

and our latest result. Also trivially $w \diamond w^2, w^2 \diamond w \neq c, w^2$. But now with (1.15)

$$w = (w^2 \diamond w) \diamond (w \diamond (w^2 \diamond w)) = (w^2 \diamond w) \diamond c = w^2 \diamond w \tag{1.17}$$

which is a contradiction.
Conclusion: The diagonal/squaring cannot be constant almost everywhere with a single exception, since such a magma doesn't even exist for the law 1323.