1 Spectrum of laws of order up to 4

1.1 First filtering

Consider laws of order up to 4, modulo duality and equivalence, and denote these 741 classes by the lowest-numbered equation:

[1, 2, 3, 4, 8, 9, 10, 11, 13, 14, 16, 38, 40, 41, 43, 47, 48, 49, 50, 52, 53, 55, 56, 58, 62, 63, 65, 66, 72, 73, 75, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 111, 115, 117, 118, 124, 125, 127, 138, 151, 152, 153, 159, 162, 167, 168, 307, 308, 309, 310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 323, 325, 326, 327, 329, 332, 333, 335, 336, 343, 411, 412, 413, 414, 416, 417, 418, 419, 420, 422, 426, 427, 428, 429, 430, 432, 433, 434, 436, 437, 439, 440, 442, 443, 446, 450, 452, 455, 463, 464, 466, 467, 473, 474, 476, 477, 481, 492, 500, 501, 503, 504, 508, 510, 511, 513, 543, 546, 556, 562, 614, 615, 616, 617, 618, 619, 620, 621, 622, 623, 624, 626, 629, 630, 632, 633, 635, 639, 640, 642, 643, 645, 646, 647, 653, 655, 657, 658, 667, 669, 670, 676, 677, 679, 680, 690, 692, 695, 703, 704, 706, 707, 713, 714, 716, 723, 727, 731, 765, 778, 817, 818, 819, 820, 822, 823, 824, 825, 826, 827, 828, 829, 832, 833, 834, 835, 836, 837, 838, 839, 840, 842, 843, 844, 845, 846, 847, 848, 854, 856, 860, 861, 870, 872, 873, 879, 880, 882, 883, 887, 895, 898, 906, 907, 910, 916, 917, 947, 960, 978, 1020, 1021, 1022, 1023, 1025, 1026, 1027, 1028, 1029, 1032, 1033, 1035, 1036, 1037, 1038, 1039, 1041, 1042, 1043, 1045, 1046, 1048, 1049, 1050, 1051, 1052, 1053, 1055, 1056, 1060, 1061, 1063, 1073, 1075, 1076, 1082, 1083, 1085, 1086, 1096, 1109, 1110, 1112, 1113, 1117, 1119, 1122, 1133, 1137, 1167, 1171, 1184, 1223, 1224, 1225, 1226, 1227, 1228, 1229, 1230, 1231, 1232, 1233, 1234, 1235, 1236, 1237, 1238, 1239, 1240, 1241, 1242, 1243, 1244, 1245, 1248, 1249, 1250, 1251, 1252, 1253, 1254, 1255, 1256, 1259, 1262, 1263, 1264, 1267, 1271, 1276, 1278, 1279, 1285, 1286, 1288, 1289, 1312, 1313, 1315, 1316, 1322, 1323, 1325, 1340, 1353, 1370, 1374, 1387, 1426, 1427, 1428, 1429, 1431, 1432, 1434, 1435, 1437, 1441, 1442, 1443, 1444, 1445, 1446, 1447, 1448, 1451, 1453, 1454, 1457, 1461, 1465, 1469, 1478, 1479, 1480, 1481, 1482, 1483, 1484, 1485, 1486, 1488, 1489, 1491, 1492, 1496, 1515, 1516, 1518, 1519, 1523, 1525, 1526, 1586, 1629, 1630, 1631, 1632, 1633, 1634, 1635, 1636, 1637, 1638, 1641, 1644, 1645, 1647, 1648, 1650, 1654, 1655, 1657, 1660, 1664, 1672, 1681, 1682, 1684, 1685, 1687, 1691, 1692, 1694, 1695, 1701, 1718, 1719, 1721, 1722, 1724, 1728, 1729, 1731, 1738, 1793, 3253, 3254, 3255, 3256, 3257, 3258, 3259, 3260, 3261, 3262, 3263, 3264, 3265, 3267, 3268, 3269, 3270, 3271, 3272, 3273, 3274, 3275, 3277, 3278, 3279, 3280, 3281, 3282, 3283, 3284, 3285, 3288, 3290, 3292, 3294, 3296, 3297, 3300, 3306, 3308, 3309, 3312, 3315, 3316, 3317, 3318, 3319, 3320, 3321, 3322, 3323, 3326, 3331, 3334, 3342, 3343, 3345, 3346, 3349, 3350, 3352, 3353, 3355, 3363, 3364, 3385, 3388, 3398, 3414, 3417, 3456, 3457, 3458, 3459, 3460, 3461, 3462, 3463, 3464, 3465, 3466, 3467, 3468, 3469, 3470, 3471, 3472, 3473, 3474, 3475, 3476, 3477, 3478, 3479, 3480, 3481, 3482, 3483, 3484, 3485, 3487, 3488, 3489, 3491, 3493, 3495, 3496, 3497, 3499, 3503, 3509, 3511, 3512, 3513, 3515, 3518, 3519, 3520, 3521, 3522, 3523, 3524, 3525, 3526, 3527, 3529, 3532, 3533, 3534, 3537, 3541, 3545, 3546, 3548, 3549, 3555, 3556, 3558, 3566, 3583, 3587, 3588, 3591, 3600, 3601, 3607, 3617, 3620, 3634, 3659, 3660, 3661, 3662, 3663, 3665, 3666, 3667, 3668, 3669, 3670, 3671, 3672, 3673, 3675, 3676, 3678, 3679, 3681, 3682, 3683, 3703, 3712, 3714, 3715, 3716, 3718, 3721, 3722, 3723, 3724, 3726, 3727, 3728, 3729, 3730, 3735, 3737, 3740, 3744, 3748, 3751, 3756, 4268, 4269, 4270, 4271, 4272, 4273, 4274, 4275, 4276, 4277, 4278, 4279, 4280, 4283, 4284, 4286, 4287, 4288, 4290, 4291, 4293, 4295, 4296, 4297, 4299, 4300, 4301, 4303, 4304, 4305, 4314, 4315, 4318, 4320, 4321, 4325, 4327, 4331, 4343, 4358, 4362, 4364, 4369, 4380, 4381, 4382, 4383, 4384, 4385, 4386, 4387, 4388, 4389, 4390, 4391, 4392, 4393, 4396, 4397, 4398, 4399, 4400, 4401, 4402, 4403, 4404, 4405, 4406, 4407, 4408, 4410, 4411, 4412, 4413, 4415, 4416, 4417, 4421, 4423, 4424, 4428, 4430, 4433, 4434, 4435, 4437, 4438, 4439, 4441, 4443, 4444, 4445, 4447, 4448, 4449, 4456, 4458, 4460, 4461, 4470, 4471, 4474, 4476, 4478, 4481, 4482, 4484, 4485, 4490, 4497, 4502, 4512, 4513, 4515, 4517, 4519, 4520, 4526, 4531, 4535, 4541, 4544]

All laws obeyed by constant laws, or left/right projection, or by $x \diamond y = \pm x \pm y$ on $\mathbb{Z}/n\mathbb{Z}$ have full spectrum. Then we remember that some of that work was already done so we read `https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Equations.20with.20full.20spectrum/with/489967644`. For laws 1482, 1523, 1682 we have models of all sizes:

- for law 1482, the model found by Douglas McNeil is $x \diamond y = 0$ except $0 \diamond 0 = 1$ and $0 \diamond x = x \diamond 0 = x$ for $x \neq 0, 1$;
- for law 1523, $x \diamond x = 0$, $x \diamond 0 = 0 \diamond x = x$.
- for law 1682, the model found by Zoltan Kocsis on any interval $\{0, 1, \dots, n-$

1} is defined by

$$i \diamond j = \begin{cases} i & \text{if } j = 0, \\ \delta_{i=0} & \text{if } j = 1, \\ 1 + \delta_{j \text{ odd}} & \text{if } i = 0 \text{ and } j \geq 2, \\ j & \text{if } i = 1 \text{ and } j \geq 3 \text{ odd}, \\ i + \delta_{i<j} & \text{if } i, j \geq 2 \text{ and } i + j \text{ even}, \\ 2\delta_{i=1} + 0 & \text{if } i \geq 1 \text{ odd and } j = 2, \\ j - \delta_{i \geq j} & \text{if } i \geq 1 \text{ and } j \geq 3 \text{ and } i + j \text{ odd}. \end{cases} \tag{1}$$

This leaves 38 laws:

[63, 115, 467, 474, 481, 501, 546, 556, 667, 670, 677, 695, 704, 873, 880, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1323, 1480, 1483, 1485, 1486, 1489, 1496, 1516, 1526, 1685, 1692, 1719]

Many of the remaining laws, in the finite setting, imply that the magma is a quasigroup (left and right multiplications are bijective). Some pairs or triplets are equivalent: Laws [63,1692], laws [73,118,125], laws [115,880], laws [481,1496], laws [882,1323,1526]. In addition, some of them are parastrophically equivalent, namely the division operation of one obeys the other: the laws 63 and 73, and the laws 546 and 556. We are left with 32 laws:

[63, 115, 467, 474, 481, 501, 546, 667, 670, 677, 695, 704, 873, 883, 887, 895, 898, 907, 1076, 1083, 1110, 1279, 1286, 1313, 1480, 1483, 1485, 1486, 1489, 1516, 1685, 1719]

Note that in a (left-)quasigroup obeying law 467 $x = y \diamond (x \diamond (x \diamond (y \diamond y)))$, the left division operation defined by $x \diamond (x : y) = y$ obeys law 437 $x = x : (y : (y : (x : y)))$, but we cannot make use of that because the models of law 437 that we used to show the full spectrum property are left-projections, which are maximally far from being left quasigroups. The same problem occurs for law 481. Laws 474 and 501 are parastrophically equivalent to themselves. The other laws do not have a shape conducive to having a parastrophically equivalent law. We organize the discussion around four classes of laws, and the remaining ones: those related to abelian groups,

1.2 Laws from which a group structure is definable

First some laws related to abelian groups. Note that there are some more laws implying linearity, which have already been eliminated at an earlier stage by models of the form $x \diamond y = \pm x \pm y$, most crucially Tarski's law 543 defining abelian group subtraction.

546 $x = y \diamond (z \diamond (x \diamond (z \diamond y)))$ has **conjectural** spectrum (easy for an expert) consisting of all numbers whose prime decomposition only involves even powers of primes congruent to 3 mod 4. For any fixed element u, the operation $x \ominus_u y \coloneqq (x \diamond u) \diamond (u \diamond y)$ obeys the Tarski law 543, namely is group subtraction. With some more work one can show that $x \diamond y = -x + iy + c$ in some module over $\mathbb{Z}[i]$, $i = \sqrt{-1}$. This should determine the spectrum:

abelian groups such that there exists an automorphism that squares to negation. I think (can someone prove?) that the spectrum consists precisely of the numbers whose prime decomposition $n = p_1^{k_1} \dots p_m^{k_m}$ is such that for all i one has $p_i^{k_i} \equiv 0, 1, 2 \mod 4$, namely p_i or $p_i \equiv 1 \mod 4$ or $k_i \equiv 0 \mod 2$.

895 $x = y \diamond ((x \diamond z) \diamond (y \diamond z))$ has spectrum $\{2^n \mid n \geq 0\}$. It characterizes Boolean groups (abelian groups of exponent 2).

898 $x = y \diamond ((x \diamond z) \diamond (z \diamond y))$ has spectrum $\{2^n \mid n \geq 0\}$. It is at least that because law 895 implies 898. To get the converse, note that the law 898 implies, for any fixed element u, that the operation $x \oplus_t y = ((t \diamond x) \diamond (y \diamond t)) \diamond t$ obeys law 895. We can be more precise about the structure: in the Boolean group, the operation $\diamond$ can be written as $x \diamond y = ax + by + c$ with a, b being endomorphisms of the group and c a group element, such that $a = -b^3$ and $(1 + b)(1 + b + b^3)(1 + b^2 + b^3) = 0$ and $(1 + b)(1 + b^2 + b^3)c = 0$.

1.3 Laws related to semi-symmetric quasigroups 14

Next, let us discuss laws related to the semi-symmetric quasigroup law 14. Here, we call Mendelsohn quasigroup an idempotent semi-symmetric quasigroup (namely obeying laws 3 and 14). They are in one-to-one correspondence with Mendelsohn triple systems on the same set so their spectrum is $\{n \equiv 0, 1 \mod 3\} \setminus \{6\}$.

Semi-symmetric loops (magmas obeying law 14 and 40) are in one-to-one correspondence with Mendelsohn triple systems on $M \setminus \{e\}$ so their spectrum is $\{n \equiv 1, 2 \mod 3\} \setminus \{7\}$.

115 $x = y \diamond ((x \diamond x) \diamond y)$ has **conjectural** spectrum $[1, +\infty) \setminus \{2, 6\}$. The law is obeyed in Mendelsohn quasigroups so its spectrum contains all $n \equiv 0, 1 \mod 3$ except 6. There remains to prove existence of models of all sizes $n = 3k + 2 \geq 5$ (showing it for $n = 8$, n prime, and $n/2$ prime would be enough). An ATP run shows there are models at least up to order 53, in which the squaring map $S \colon x \mapsto x \diamond x$ has a large cycle of size $3k + 1$ (or possibly $3j + 1$ for j close to k), making most of the magma translationally-invariant. For magma sizes up to 14, the possible cycle sizes for the squaring map (apart from having it be the identity) are $5 = 4 + 1$, $7 = 7$, $8 = 7 + 1$, $9 = 6 + 1 + 1 + 1$, $10 = 7 + 1 + 1 + 1$, $11 = 5 + 5 + 1$, $11 = 7+1+1+1+1$, $12 = 8+1+1+1+1$, $12 = 9+1+1+1$, $13 = 4+4+4+1$, $13 = 8 + 4 + 1$, $13 = 9 + 1 + 1 + 1 + 1$, $13 = 13$, $14 = 5 + 5 + 1 + 1 + 1 + 1$, $14 = 7 + 7$, $14 = 10 + 1 + 1 + 1 + 1$, $14 = 12 + 1 + 1$. There is probably some counting argument for which sizes are possible.

481 $x = y \diamond (x \diamond (y \diamond (z \diamond z)))$ has **conjectural** spectrum $[1, +\infty) \setminus \{3, 6\}$. It is equivalent to unipotence $x \diamond x = e$ and $x = y \diamond (x \diamond (y \diamond e))$. Finite magmas are quasigroups and the maps $x \mapsto x \diamond e$ and $x \mapsto e \diamond x$ are automorphisms, inverses of each other. The special case where these automorphisms are the

identity is semi-symmetric loops. The spectrum thus contains $\{n \equiv 1, 2 \mod 3\} \setminus \{7\}$. It also contains $\{7, 9, 12\}$ by an ATP search. We see in these examples that the squaring map has specific cycle sizes. A natural conjecture is that the law is flexible enough to allow arbitrary magma sizes, except for the low-lying values 3,6.

501 $x = y \diamond (y \diamond (x \diamond (x \diamond y)))$ has **unknown** spectrum containing 1,4,5,8,9. It implies that the operation $x \Box y = x \diamond (x \diamond y)$ obeys law 14.

667 $x = y \diamond (x \diamond ((x \diamond x) \diamond y))$ has **unknown** spectrum containing $\{n \equiv 1, 2 \mod 3\} \cup \{9\} = \{1, 2, 4, 5, 7, 8, 9, 10, 11, \dots\}$. It is obeyed by semi-symmetric loops (hence the spectrum contains $\{n \equiv 1, 2 \mod 3\}$ except 7, and by idempotent magmas satisfying the Dupont law 63 (see below), from which one gets 7 in the spectrum. An ATP run gives that 9 is also there.

695 $x = y \diamond (x \diamond ((z \diamond z) \diamond y))$ has spectrum $\{n \equiv 1, 2 \mod 3\} \setminus \{7\}$. Indeed, any semi-symmetric loop obeys law 695, and conversely the operation $x \Box y = (x \diamond x) \diamond (x \diamond y)$ obeys the semi-symmetric loop law 887.

873 $x = y \diamond ((x \diamond x) \diamond (y \diamond y))$ **conjectural** spectrum $[1, +\infty) \setminus \{2, 6\}$. The spectrum contains that of law 115 because the dual of law 115 implies law 873. The spectrum does not contain 2 and 6 thanks to an ATP run.

883 $x = y \diamond ((x \diamond y) \diamond (y \diamond y))$ has **unknown** spectrum containing $\{1, 2 \mod 3\}$. It is obeyed by semi-symmetric loops *and* idempotent magmas obeying law 73 (parastrophically equivalent to the Dupont law). The size 7 model is such an idempotent magma.

887 $x = y \diamond ((x \diamond y) \diamond (z \diamond z))$ has spectrum $\{n \equiv 1, 2 \mod 3\} \setminus \{7\}$. Indeed, the law characterizes exactly the semi-symmetric loops, equivalent to Mendelsohn triple systems on $M \setminus \{e\}$.

1083 $x = y \diamond ((x \diamond (y \diamond x)) \diamond y)$ has **unknown** spectrum containing 1,3,4,7,8,9. It implies that $x \Box y = x \diamond (y \diamond x)$ obeys law 14.

1719 $x = (y \diamond y) \diamond ((x \diamond x) \diamond y)$ has **unknown** spectrum containing $\{0, 1 \mod 3\} \cup \{5, 8\}$, possibly $[1, +\infty) \setminus \{2\}$? Indeed, it is obeyed by Mendelsohn quasigroups so the spectrum contains $\{0, 1 \mod 3\} \setminus \{6\}$, and magmas of size $5, 6, 8$ are easily found by an ATP.

1.4 Twists of the Dupont law 63

Next, we can discuss laws related to the Dupont law 63 and to law 73, its parastrophic equivalent.

63 $x = y \diamond (x \diamond (x \diamond y))$ has **unknown** spectrum starting with $1, 3, 4, 5, 7, 8, 9, 11, 12, 13$ and containing most sizes $n \equiv 0, 1, 3 \mod 4$ up to 100. Many magmas (including some of size 9 that I think are non-linear) obey $x = y \diamond (y \diamond (y \diamond x))$ in addition to law 63.

467 $x = y \diamond (x \diamond (x \diamond (y \diamond y)))$ has **unknown** spectrum starting with $1, 5, 7, 8$. It is the Dupont law twisted by the squaring map.

704 $x = y \diamond (y \diamond ((x \diamond x) \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is a twist of law 73 by the squaring map.

1110 $x = y \diamond ((y \diamond (x \diamond x)) \diamond y)$ has **unknown** spectrum starting with $1, 4, 5, 7, 8, 9$. It is a twist of law 73 by the squaring map.

1279 $x = y \diamond (((x \diamond x) \diamond y) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7, 8$ and no 9. It is (dual to) a twist of law 73 by the squaring map.

1516 $x = (y \diamond y) \diamond (x \diamond (x \diamond y))$ has **unknown** spectrum starting with $1, 5, 7, 8$. It is the Dupont law twisted by the squaring map.

1.5 Specializations of the central groupoid law 168

Some laws implied by the central groupoid law 168 hence their spectrum contains $\{n^2 \mid n \geq 1\}$.

1480 $x = (y \diamond x) \diamond (x \diamond (x \diamond z))$ has **conjectural** spectrum $\{1\} \cup [4, +\infty)$, checked with an ATP up to order 18. Experimentally, it is even possible to take $x \diamond y = 0$ for $\lfloor \sqrt{n} \rfloor$ different values of y (different from 0).

1483 $x = (y \diamond x) \diamond (x \diamond (y \diamond z))$ has **unknown** spectrum starting with $1, 2, 4, 8, 9$.

1485 $x = (y \diamond x) \diamond (x \diamond (z \diamond y))$ has **conjectural** spectrum $\{n^2 \mid n \geq 1\} \cup \{2n^2 \mid n \geq 1\}$ as discussed at `https://leanprover.zulipchat.com/#narrow/stream/458659-Equational/topic/1485`.

1486 $x = (y \diamond x) \diamond (x \diamond (z \diamond z))$ has **unknown** spectrum containing squares and $\{n^2+2 \mid n \geq 3\} \cup \{13, 21\}$, discussed at `https://leanprover.zulipchat.com/#narrow/channel/458659-Equational/topic/Understanding.20Finite.201486.20Magmas`

Some laws seem unrelated.

474 $x = y \diamond (x \diamond (y \diamond (x \diamond y)))$ has spectrum $[1, +\infty) \setminus \{2, 4\}$. It has models of all odd sizes $n = 2k + 1$, given by (for $0, x, y$ distinct) $0 \diamond 0 = 0$, $x \diamond x = 0$, $x \diamond 0 = x$, $0 \diamond x = \sigma(x)$ an involution without fixed point, $x \diamond y = y$. It has a model of all even sizes ≥ 6 given by an explicit multiplication table for $0 \leq x, y \leq 5$,

	0	1	2	3	4	5
0	0	2	3	4	5	1
1	1	0	5	4	3	2
2	2	3	0	1	5	4
3	3	5	4	0	2	1
4	4	2	1	5	0	3
5	5	4	3	2	1	0

(2)

together with $x \diamond x = 0$ and $x \diamond 0 = x$ for all x, and $0 \diamond x = \sigma(x)$ an involution without fixed point for $6 \leq x < n$, and $x \diamond y = y$ for all remaining entries, namely $\min(x, y) \geq 1$ and $\max(x, y) \geq 6$ and $x \neq y$.

670 $x = y \diamond (x \diamond ((x \diamond y) \diamond y))$ has **unknown** spectrum starting with $1, 4, 5$, not 6 nor 7

677 $x = y \diamond (x \diamond ((y \diamond x) \diamond y))$ has **unknown** spectrum. It is the source of the famous last surviving implication.

907 $x = y \diamond ((y \diamond x) \diamond (x \diamond y))$ has **unknown** spectrum containing $1, 3, 7, 9, 13$, and not $2, 4, 5, 6$. Models tend to either have a left identity or to be commutative and idempotent.

1076 $x = y \diamond ((x \diamond (x \diamond y)) \diamond y)$ has **unknown** spectrum starting with $1, 5$, and not having 6 nor 7.

1286 $x = y \diamond (((x \diamond y) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 7$.

1313 $x = y \diamond (((y \diamond x) \diamond x) \diamond y)$ has **unknown** spectrum starting with $1, 5, 7$.

1489 $x = (y \diamond x) \diamond (y \diamond (x \diamond y))$ has **conjectural** spectrum $[1, +\infty) \setminus \{2, 4\}$, starting with $1, 3, 5, 6, 7, 8, 9$, many of which are idempotent and with $x \diamond y = y$ for many entries.

1685 $x = (y \diamond x) \diamond ((x \diamond y) \diamond y)$ has **unknown** spectrum starting with $1, 3, 4, 5, 6, 7$.