Nets

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March 7, 2019

I bet, with my net,
I can get those things yet.
Dr. Seuss.

(Note written 29/4/14)

1 Introduction

I was motivated to learn something about topological vector spaces, and in particular I wanted to understand what it meant to say that a topological vector space was complete. In this generality one wants to ask more than all Cauchy sequences converging, because one does not know that the topology is given by a metric. The correct generalisation of a sequence in this context is a *net*, so here are some basic things about nets. All of this is taken from "Real analysis" by G. Folland. In particular, I am just giving definitions and proving or stating basic results from this book throughout. I should perhaps remark that I discovered after writing this that for first countable topological vector spaces (see definition below—but note that Fréchet spaces are metrisable and hence first countable), completeness really *is* just asking that all Cauchy sequences converge (see Chapter 5, Exercise 44 of Folland), so a lot of this nonsense below is not necessary in many cases. It's still pretty neat though, and in particular after over 15 years I finally understand why Ray Lickorish only proved that compactness was equivalent to sequential compactness for metric spaces in Part IB.

2 Why nets?

Recall that if X is a topological space, and $x_1, x_2, \dots$ is a sequence of elements of X, then we say that $(x_i)_i \to x$ for $x \in X$ if, for all neighbourhoods A of x, there is $N \geq 0$ such that $x_i \in A$ for all $i \geq N$. Looks like a nice idea. But there is a problem with this notion of convergence in the generality of topological spaces, that you don't see for metric spaces. The problem is this. If X is a topological space, and A is a subset, then the closure $\overline{A}$ of A in X may be strictly bigger than the set of limits (in X) of sequences all of whose elements are in A. If x is not in $\overline{A}$ then certainly it's not a limit of a sequence all of whose terms are in A. But if X is sufficiently "big" then there could be elements in $\overline{A}$ which are also not limits of sequences in A. More precisely, we say that a topological space X is *first countable* if every point has a countable neighbourhood base, that is, for all $x \in X$ there is a countable set U_i of open neighbourhoods of x with the property that every open subset of X containing x also contains one of the U_i. Metric spaces are first countable (balls of rational radius), and sequences are a good idea in metric spaces. On the other hand, for spaces that are not first countable, the two "natural" notions of closure above may not coincide, as we shall now see (this example is on p125 of Folland).

The set $\mathbf{C}^{\mathbf{R}}$ of (not necessarily continuous) functions from $\mathbf{R}$ to $\mathbf{C}$ is a product of copies of $\mathbf{C}$, and if one gives it the product topology, then one checks easily that for functions $(f_n)_{n\geq 0}$ and f, we have $f_n \to f$ iff $f_n(x) \to f(x)$ for all x (so convergence in the product is just pointwise convergence—see Proposition 4.12 of Folland, or reconstruct the argument yourself). Now consider the subset A of continuous functions $\mathbf{R} \to \mathbf{C}$. Recall that a function $\mathbf{R} \to \mathbf{C}$ is *Borel measurable* if the pre-image of a Borel set is Borel (note that this is stronger than Lebesgue measurability), and continuous functions are Borel measurable, and a pointwise limit of Borel measurable functions is Borel measurable (Folland Corollary 2.9), so the set of limits of sequences of elements of A are contained within the Borel measurable functions. On the other hand, A is dense in $\mathbf{C}^{\mathbf{R}}$ because if $f : \mathbf{R} \to \mathbf{C}$ and U is an open set in $\mathbf{C}^{\mathbf{R}}$ containing f then, after shrinking U if necessary, there exists $\varepsilon > 0$ and $x_1, x_2, \ldots, x_n \in \mathbf{R}$ such that U is the functions $g : \mathbf{R} \to \mathbf{C}$ such that $|g(x_i) - f(x_i)| < \varepsilon$ for all i, and visibly there will be a continuous function with this property.

So how does one recover the closure of a set A in a topological space X as a set of "limits"? The answer is to use nets.

3 Nets

Definitions: a *directed set* is a set S equipped with a binary relation $\leq$ satisfying $s \leq s$ for all $s \in S$, $s \leq t \leq u$ implies $s \leq u$, and for any $s, t \in S$ there is $u \in S$ with $s \leq u$ and $t \leq u$. Examples are the natural numbers with the usual $\leq$, and the neighbourhoods of a point x in a topological space X, with $U \leq V$ iff $V \subseteq U$ (reverse inclusion). A *net* in a topological space X is a map $S \to X$ with S a directed set. Notation: $s \in S$ is sent to $x_s \in X$. If $(x_s)_{s \in S}$ is a net and $E \subseteq X$ is a subset, we say that the net is *eventually in* E if there exists $s_0 \in S$ such that $x_s \in E$ for all $s \geq s_0$ (This implies, but I suspect is stronger than, demanding that for all $s \in S$ there is $t \geq s$ with $x_t \in E$), and we say that $x \in X$ is a *limit* of $(x_s)_{s \in S}$, or (x_s) *converges to* x, or just $x_s \to x$, if for every neighbourhood U of x, (x_s) is eventually in U.

The result is that if $E \subseteq X$ then $x \in \overline{E}$ iff there is a net in E that converges to x. This is Proposition 4.18 of Folland. Here's a proof: it's clear that if x is not in the closure then it's certainly not a limit, so every limit is in the closure. Conversely, if x is in the closure, then let S be the neighbourhoods of x ordered by reverse inclusion, and for $s \in S$ define x_s to be an element of $E \cap s$. The claim is that this net converges to x and this is clear because if U is a neighbourhood of x then we set $s_0 = U$ and for $s \geq s_0$ we have $x_s \in s \subseteq U$.

Remark: if $f : X \to Y$ is a map of topological spaces, then f is continuous at $x \in$ iff for every net (x_s) converging to x, $f(x_s)$ converges to $f(x)$. This is Proposition 4.19 of Folland, I'll omit the proof though.

Recall from my 2nd year analysis notes that a metric space is compact iff it's sequentially compact (that is, if every sequence has a convergent subsequence). The correct generalisation is that a topological space is compact iff every net has a cluster point iff every net has a convergent subnet, but I will not define these terms: see p126 of Folland.